

FAST MAGNETIC RECONNECTION AND SPONTANEOUS STOCHASTICITY

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ABSTRACT

Magnetic field-lines in astrophysical plasmas are expected to be frozen-in at scales larger than the ion gyroradius. The rapid reconnection of magnetic flux structures with dimensions vastly larger than the gyroradius requires a breakdown in the standard Alfvén flux-freezing law. We attribute this breakdown to ubiquitous MHD plasma turbulence with power-law scaling ranges of velocity and magnetic energy spectra. Lagrangian particle trajectories in such environments become “spontaneously stochastic”, so that infinitely-many magnetic field-lines are advected to each point and must be averaged to obtain the resultant magnetic field. The relative distance between initial magnetic field lines which arrive to the same final point depends upon the properties of two-particle turbulent dispersion. We develop predictions based on the phenomenological Goldreich & Sridhar theory of strong MHD turbulence and on weak MHD turbulence theory. We recover the predictions of the Lazarian & Vishniac theory for the reconnection rate of large-scale magnetic structures. Lazarian & Vishniac also invoked “spontaneous stochasticity”, but of the field-lines rather than of the Lagrangian trajectories. More recent theories of fast magnetic reconnection appeal to microscopic plasma processes that lead to additional terms in the generalized Ohm’s law, such as the collisionless Hall term. We estimate quantitatively the effect of such processes on the inertial-range turbulence dynamics and find them to be negligible in most astrophysical environments. For example, the predictions of the Lazarian-Vishniac theory are unchanged in Hall MHD turbulence with an extended inertial range, whenever the ion skin depth δ_i is much smaller than the turbulent integral length or injection-scale L_i .

Subject headings: galaxies: magnetic fields – methods: theoretical – MHD – turbulence

1. INTRODUCTION

It is generally believed that magnetic field embedded in a highly conductive fluid preserves its topology for all time due to magnetic fields being frozen-in (Alfvén 1942; Parker 1979). Although ionized astrophysical objects, like stars and galactic disks, are almost perfectly conducting, they show indications of changes in topology, “magnetic reconnection”, on dynamical time scales (Parker 1970; Lovelace 1976; Priest & Forbes 2002). Reconnection can be observed directly in the solar corona (Innes et al. 1997; Yokoyama & Shibata 1995; Masuda et al. 1994), but can also be inferred from the existence of large-scale dynamo activity inside stellar interiors (Parker 1993; Ossendrijver 2003). Solar flares usually (Sturrock 1966) and γ -ray bursts often (Zhang & Yan 2011; Fox et al. 2005; Galama et al. 1998) are associated with magnetic reconnection. Much recent work

has concentrated on showing how reconnection can be rapid in plasmas with very small collisional rates (Shay et al. 1998; Drake 2001; Drake et al. 2006; Daughton et al. 2006), which substantially constrains astrophysical applications of the corresponding reconnection models.

Reconnection occurs rapidly in computer simulations due to the high values of resistivity (or numerical resistivity) that are employed at the low resolutions currently achievable. Therefore, if there are situations where magnetic fields reconnect slowly, numerical simulations do not adequately reproduce astrophysical reality. This means that if collisionless reconnection is the only way to make reconnection rapid, then the numerical simulations of many astrophysical processes, including those in interstellar media (ISM), which is collisional, are in error. At the same time, it is not possible just to claim that the reconnection must always be rapid empirically, as solar flares require periods of flux accumulation time, which correspond to slow reconnection.

To understand the difference between reconnection in astrophysical situations and in numerical simulations, one should recall that the dimensionless combination that controls the reconnection rate is the Lundquist number¹, defined as $S = L_x v_A / \lambda$, where L_x is the length of the reconnection layer, v_A is the Alfvén velocity, and $\lambda = \eta c^2 / 4\pi$ is Ohmic diffusivity. Because of the huge astrophysical length-scales L_x involved, the astrophysical Lundquist numbers are also huge, e.g. for the ISM they are about 10^{16} , while present-day MHD simulations correspond to $S < 10^4$. As the simulation costs scale as L_x^4 , where L_x is the size of the box, it is feasible neither at present nor in the foreseeable future to have simulations with sufficiently high Lundquist numbers.

Plasma flows at such high Lundquist numbers are generically turbulent, since laminar flows are then prey to numerous linear and finite-amplitude instabilities. Indeed, turbulence is ubiquitous in astrophysical plasmas. This is sometimes driven turbulence due to an external energy source, such as supernova in the ISM (Norman & Ferrara 1996; Ferrière 2001), merger events and AGN outflows in the intercluster medium (ICM) (Subramanian et al. 2006; Enßlin & Vogt 2006; Chandran 2005), and baroclinic forcing behind shock waves in interstellar clouds. In other cases, the turbulence is spontaneous, with available energy released by a rich array of instabilities, such as MRI in accretion disks (Balbus & Hawley 1998), kink instability of twisted flux tubes in the solar corona (Galsgaard & Nordlund 1997a; Gerrard & Hood 2003), etc. Whatever its origin, the signatures of plasma turbulence are seen throughout the universe. Turbulent cascade of energy leads to long “inertial ranges” with power-law spectra that are widely observed, e.g. in the solar wind (Leamon et al. 1998; Bale et al. 2005), in the ISM (Armstrong et al. 1995; Chepurnov & Lazarian 2010), and in the ICM (Schuecker et al. 2004; Vogt & Enßlin 2005). Often inertial-range spectra cannot be directly observed, but only large-scale turbulent fluctuations, e.g. through their Doppler broadening of line spectra. Nevertheless, the power-law ranges are universal features of high-Reynolds-number turbulence which, even when not seen, can be inferred to be present from enhanced rates of dissipation and mixing (Eyink 2008). Any theory of astrophysical reconnection must take into account the pre-existing turbulent environment. Even if the plasma flow is initially laminar, kinetic energy release by reconnection due to some slower plasma process may generate vigorous turbulent motion.

But can turbulence itself accelerate reconnection? Lazarian & Vishniac (1999) [henceforth LV99] proposed a model of fast reconnection in the presence of sub-Alfvénic turbulence in magnetized plasmas. They identified stochastic wandering of magnetic field-lines as the most critical property of MHD turbulence which permits fast reconnection. As we discuss more below, this line-wandering widens the outflow region and alleviates the controlling constraint of mass conservation. The LV99 model has been successfully tested recently in Kowal et al. (2009) (see also higher resolution results in Lazarian

et al. (2010)). The model is radically different from its predecessors which also appealed to the effects of turbulence (see more comparisons in section 5.4). For instance, unlike Speiser (1970) and Jacobson & Moses (1984) the model does not appeal to changes of microscopic properties of plasma. The nearest progenitor to LV99 was the work of Matthaeus & Lamkin (1985, 1986), who studied the problem numerically in 2D MHD and who suggested that magnetic reconnection may be fast due to a number of turbulence effects, e.g. multiple X points and turbulent EMF. However, Matthaeus & Lamkin (1985, 1986) did not observe the important role of magnetic field-line wandering, and did not obtain a quantitative prediction for the reconnection rate, as did LV99.

The success of LV99 in identifying an MHD turbulence mechanism for fast reconnection leads, however, to a conflict with certain conventional beliefs. As the predicted reconnection velocity is independent of magnetic diffusivity λ , the LV99 theory implies that field-line topology should change in MHD plasmas at a finite rate even in the limit of infinite Lundquist number. This contradicts the accepted wisdom that magnetic field-lines should be “nearly” frozen-in to very high-conductivity MHD plasmas. It is implicit in the LV99 theory that the standard Alfvén Theorem on magnetic-flux conservation must be violated for $\lambda \rightarrow 0$ (Vishniac & Lazarian 1999).

In a separate, more recent development, Eyink (2011) has critically examined the concept of frozen-in field-lines for turbulent MHD plasmas. See also Eyink & Aluie (2006); Eyink (2007, 2009). A first contribution of that work was to establish exact flux-conservation properties for *resistive* MHD solutions. The magnetic field at a point was shown to result from averaging over an infinite ensemble of line-vectors stochastically advected to that point by the fluid velocity perturbed with a Brownian motion proportional to $\sqrt{\lambda}$. For smooth, laminar flows the contribution of the random Brownian term disappears as $\lambda \rightarrow 0$ and standard flux-freezing is recovered. However, it was shown in Eyink (2011) that flux-conservation may remain stochastic even at infinite conductivity for rough, turbulent velocities. This remarkable behavior is due to “spontaneous stochasticity” of Lagrangian fluid particles undergoing turbulent two-particle or Richardson diffusion: see Bernard et al. (1998); Gawędzki & Vergassola (2000); E & vanden Ei-jnden (2000, 2001); Chaves et al. (2003), and, for a review, Kupiainen (2003). The work of Eyink removes the objection to LV99 based on the Alfvén theorem for high conductivity plasmas, since that law is fundamentally altered in a turbulent flow and poses no necessary obstacle to fast reconnection. Indeed, there are very close relations between the work of Eyink and LV99. For example, it was pointed out in Eyink & Aluie (2006) that the “stochastic wandering” of field-lines in a rough magnetic field which was invoked in LV99 is exactly analogous to the “spontaneous stochasticity” phenomenon for Lagrangian particle trajectories guided by a rough, turbulent velocity field.

In this work we shall establish even deeper connections between the work of Eyink and LV99. In particular, we show that the detailed, quantitative predictions of the LV99 theory can be rederived by applying the stochastic flux-freezing laws of Eyink (2011) to the problem

¹ The magnetic Reynolds number, which is the ratio of the magnetic field decay time to the eddy turnover time, is defined using the injection velocity v_l as a characteristic speed instead of the Alfvén speed v_A , which is taken in the Lundquist number.

of magnetic reconnection. In this derivation, the LV99 theory emerges as the natural turbulent analogue of the Sweet-Parker laminar solution, with resistive diffusion of field-lines through the reconnection layer replaced by turbulent Richardson diffusion. To estimate the latter, we employ the same phenomenological turbulence model as did LV99, the Goldreich & Sridhar (1995) theory [hereafter, GS95], although alternative theories would lead to similar results. The new derivation demonstrates the dynamical consistency of the LV99 theory and also eliminates the need for some difficult arguments developed in LV99 concerning the “motion” of already-reconnected field-lines.

A second main purpose of this paper is to explain in a clear physical manner the concept of “spontaneous stochasticity”, both of Lagrangian trajectories and of magnetic field-lines, and the related notion of stochastic flux-freezing. “Spontaneous stochasticity” has some rather dramatic implications, such as the breakdown of Laplacian determinism for classical dynamics. Nevertheless, the phenomenon is not difficult to understand in intuitive terms. We shall also try to explain the concept of stochastic flux-freezing in high (kinetic and magnetic) Reynolds-number MHD turbulence in the simplest possible manner, explicating some points that were left implicit in Eyink (2011). An essential point which has been emphasized many times before (Newcomb 1958; Vasyliunas 1972; Alfvén 1976) is that the notion of flux-freezing is largely conventional. While it is a very useful intuitive device, the frozenness of field-lines is a “meta-physical” principle which cannot be subjected to direct experimental test. Any definition of field-line motion that is consistent with the hydromagnetic equations is equally acceptable. We believe that the stochastic approach to flux-freezing will provide a powerful heuristic tool in astrophysics. In particular, there should be important applications whenever the plasma fluid velocity field has an extended inertial-range and the stochastic motion of field-lines is therefore due to turbulent advection rather than to resistivity. This paper presents an illustrative “case study” of an application to the problem of large-scale turbulent reconnection, but many other uses can be envisaged. As another example, Eyink & Neto (2010), Eyink (2010, 2011) discuss the critical role of stochastic flux-freezing to the mechanism of the turbulent kinematic dynamo for magnetic Prandtl numbers either small or unity.

The detailed contents of this paper are as follows: In §2 we briefly recall the justification for an MHD description of astrophysical plasmas. In §3 we discuss the scaling laws of MHD turbulence and Lagrangian particle dynamics in MHD turbulence, and introduce the concepts of stochastic flux freezing & spontaneous stochasticity. We next discuss the generalized Ohm’s law in §4. Using the concept of spontaneous stochasticity, we rederive the predictions of LV99 theory of turbulent reconnection in §5 and consider various limiting cases. We provide a discussion of our results in §6 and a final summary in §7.

2. VALIDITY OF A MAGNETOHYDRODYNAMIC DESCRIPTION

An MHD (or nearly MHD) description of a plasma can be justified based either on collisionality or on strong magnetization. This issue has been well-studied and the

subject of many discussions, e.g. Kulsrud (1983), so we shall here just briefly review the standard understanding. There are three characteristic length-scales of importance: the ion gyroradius ρ_i , the ion mean-free-path length $\ell_{mfp,i}$ arising from Coulomb collisions, and the scale L of large-scale variation of magnetic and velocity fields. Astrophysical plasmas are in many cases “strongly collisional” in the sense that $\ell_{mfp,i} \ll \rho_i$, e.g. the interiors of stars and accretion disks. In such cases, a standard Chapman-Enskog expansion yields a fluid description of the plasma (Braginsky 1965), either a two-fluid model for scales between $\ell_{mfp,i}$ and the ion skin-depth $\delta_i = \rho_i/\sqrt{\beta_i}$ or simple MHD at scales much larger than δ_i . However, astrophysical plasmas are often at most only “weakly collisional” with $\ell_{mfp,i} \gg \rho_i$, especially when they are very hot and diffuse. This can be seen from the simple relation

$$\frac{\ell_{mfp,i}}{\rho_i} \propto \frac{\Lambda}{\ln \Lambda} \frac{v_A}{c}, \quad (1)$$

which follows from the standard formula for the Coulomb collision frequency (e.g. see Fitzpatrick (2011), eq.(1.25)). Here $\Lambda = 4\pi n\lambda_D^3$ is the plasma parameter, or number of particles within the Debye screening sphere. When astrophysical plasmas are very weakly coupled (hot and rarified), then Λ is extremely large. Of course, $v_A < c$ as well, but the ratio v_A/c is generally only moderately small in well-magnetized plasmas. Typical values for some weakly coupled cases are shown in the following table:

Table 1
Representative Parameters for Some Weakly-Coupled Astrophysical Plasmas

Parameter	warm ionized ISM ^a	post-CME current sheets ^b	solar wind at magnetosphere ^c
density n , cm^{-3}	.5	7×10^7	10
temperature T , eV	.7	10^3	10
plasma parameter Λ	4×10^9	2×10^{10}	5×10^{10}
ion thermal velocity $v_{th,i}$, cm/s	10^6	3×10^7	5×10^6
ion mean-free-path $\ell_{mfp,i}$, cm	6×10^{11}	10^{10}	7×10^{12}
magnetic diffusivity λ , cm^2/s	10^7	8×10^2	6×10^5
magnetic field B , G	10^{-6}	1	10^{-4}
plasma beta β	14	3	1
Alfvén speed v_A , cm/s	3×10^5	3×10^7	7×10^6
ion gyroradius ρ_i , cm	10^8	3×10^3	6×10^6
large-scale velocity U , cm/s	10^6	4×10^6	5×10^6
large length scale L , cm	10^{20}	5×10^{10}	10^8
Lundquist number $S_L = \frac{v_A L}{\lambda}$	3×10^{18}	2×10^{15}	10^9
resistive length* ℓ_η^\perp , cm	5×10^5	1	20

^aNorman & Ferrara (1996); Ferrière (2001) ^bBemporad et al. (2008) ^cZimbaro et al. (2010)

*This nominal resistive scale is calculated from $\ell_\eta^\perp \simeq L(v_A/U)S_L^{-3/4}$, assuming GS95 turbulence holds down to that scale (see eq.(72)), and should not be taken literally when $\ell_\eta^\perp < \rho_i$.

The primary expansion to justify a hydrodynamic description of these weakly coupled but well-magnetized plasmas is an expansion in the small ion gyroradius. This yields at all length scales larger than ρ_i a description by “kinetic MHD equations”, nearly identical to standard MHD. The most significant difference is that the pressure tensor in the momentum equation is anisotropic, with the two components p_\parallel and p_\perp of the pressure parallel and perpendicular to the local magnetic field direction determined from a 1-dimensional kinetic equation (Kulsrud 1983). The magnetic field solves an ideal induction equation, if one ignores all collisional effects,

although electron-ion collisions produce in reality a non-zero resistivity η . However, in very many cases (cf. the ISM and the magnetosphere in Table 1) the resistive length-scale ℓ_η^\perp is much smaller than ρ_i and also than $\rho_e \approx \frac{1}{43}\rho_i$. Magnetic field-lines are, at least formally, well “frozen-in” on these scales. Plasmas that are not strongly collisional further divide into two cases: “collisionless” plasmas for which $\ell_{mfp,i} \gg L$, the largest scales of interest, and “weakly collisional” plasmas for which $L \gg \ell_{mfp,i}$. In the latter case the “kinetic MHD” description can be further reduced in complexity at scales greater than $\ell_{mfp,i}$ by including the Coulomb collision operator and making again a Chapman-Enskog expansion. This reproduces a fully hydrodynamic MHD description at those scales, with anisotropic transport behavior associated to the well-magnetized limit. Among our examples in Table 1 above, the warm ionized ISM is “weakly collisional”, while post-CME current sheets and the solar wind impinging on the magnetosphere are (nearly) “collisionless.”

Additional important simplifications occur if the following assumptions are satisfied: turbulent fluctuations are small compared to the mean magnetic field, have length-scales parallel to the mean field much larger than perpendicular length-scales, and have frequencies low compared to the ion cyclotron frequency. These are standard assumptions of the Goldreich-Sridhar (GS95) theory of MHD turbulence, discussed in the following section, but are often valid much more generally in astrophysical plasmas. They are the basis of the “gyrokinetic approximation” which was developed for fusion plasmas but more recently extensively applied in astrophysics (Schekochihin et al. 2007; Schekochihin et al. 2009). At length-scales greater than the ion gyro-radius ρ_i , which mainly concern us in this work, a remarkable reduction occurs. The incompressible shear-Alfvén wave modes have an autonomous dynamics unaffected by the compressive modes and described by the simple “reduced MHD” (RMHD) equations. This fact is of fundamental importance for the theory developed in the present work, permitting us to base our analysis on an incompressible MHD fluid model. Compressible fast magnetosonic waves are subject to strong damping both from collisionless effects and in shocks, and, also, are not regenerated by the incompressible dynamics (see below). The other compressible plasma modes (slow and entropy waves) are “passively” transported by the shear-Alfvén waves. For “collisionless” plasmas the compressible modes are described by gyrokinetic equations for a distribution function g in phase space and dissipated by collisionless damping. For “weakly collisional” plasmas at scales greater than $\ell_{mfp,i}$ the slow and entropy waves are described by the compressible MHD equations and damped primarily by (field-parallel) viscosity and thermal diffusion.

Although we are generally interested in this work in reconnection phenomena at length scales much larger than ρ_i , there are some situations where the scales of interest are comparable to the ion gyroradius, e.g. in the magnetosphere (see Table 1). At scales smaller than ρ_i but larger than ρ_e , the plasma is described by an ion kinetic equation and a system of “electron reduced MHD” (ERMHD) equations for kinetic Alfvén waves (Schekochihin

et al. 2007; Schekochihin et al. 2009). This system exhibits the “Hall effect”, with distinct ion and electron mean flow velocities and magnetic field-lines frozen-in to the electron fluid. The ERMHD equations (or the more general “electron MHD” or EMHD equations) produce the typical features of “Hall reconnection” such as quadrupolar magnetic fields in the reconnection zone (Uzdensky & Kulsrud 2006).² At length scales smaller than ρ_e , kinetic equations are required to describe both the ions and the electrons. It is at these scales that the magnetic flux finally “unfreezes” from the electron fluid, due to effects such as Ohmic resistivity, electron inertia, finite electron gyroradius, etc. However, as we shall discuss at length in this work, these weak effects are vastly accelerated by turbulent advection and manifested, in surprising ways, at far larger length scales.

We end this short review with just a few additional references. The effects of compressible modes have been extensively studied in numerical simulations of isothermal compressible MHD by Cho & Lazarian (2002, 2003) (see also Kowal & Lazarian (2010)). Separating MHD fluctuations into Alfvén, slow and fast modes, they showed that the energy transfer from Alfvénic to compressible modes scales $\propto (\delta v)^2/(v_A^2 + c_s^2)$, where δv is the velocity perturbation at the given scale and v_A and c_s are Alfvén and sound speeds, respectively. As a result, for many astrophysically important cases the energy exchange is suppressed to a high degree. The Alfvén-wave cascade then decouples from the compressible degrees of freedom and the shear-Alfvén modes in the compressible regime exhibit scalings and anisotropy very similar to those in the incompressible regime. These results give further justification in our present study to adopting a simple model of an incompressible MHD plasma. As we shall see, it is the shear-Alfvén modes that are responsible for all the principal effects discussed below. Finally, Cho & Lazarian (2004) have carried out a numerical study of three-dimensional EMHD turbulence. They observed a critically-balanced, anisotropic EMHD cascade quantitatively different, but qualitatively the same, as the GS95 strong MHD turbulence discussed in the following section. Although we focus in this work on reconnection phenomena at scales much larger than ρ_i , similar ideas may apply at scales less than ρ_i in the EMHD regime.

² Because the Hall MHD equations have played a prominent role in magnetic reconnection research of the past decade (Shay et al. 1998, 1999; Wang et al. 2000; Birn et al. 2001; Drake 2001; Malakit et al. 2009; Cassak et al. 2010), it is worth remarking that those equations are essentially never applicable in astrophysical environments. A derivation of Hall MHD based on collisionality requires that the ion skin-depth δ_i must satisfy the conditions $\delta_i \gg L \gg \ell_{mfp,i}$. The second inequality is needed so that a two-fluid description is valid at the scales L of interest, while the first inequality is needed so that the Hall term remains significant at those scales. However, substituting $\delta_i = \rho_i/\sqrt{\beta_i}$ into (1) yields the result

$$\frac{\ell_{mfp,i}}{\delta_i} \propto \frac{\Lambda}{\ln \Lambda} \frac{v_{th,i}}{c}.$$

The ratio $v_{th,i}/c$ is generally small in astrophysical plasmas, but the plasma parameter Λ is usually large by even much, much more (see Table 1). Thus, it is usually the case that $\ell_{mfp,i} \gg \delta_i$, unless the ion temperature is extremely low. A collisionless derivation of Hall MHD from gyrokinetics requires also a restrictive condition of cold ions (Schekochihin et al. (2009), Appendix E). Thus, Hall MHD is literally valid only for cold, dense plasmas like those produced in some laboratory experiments, such as the MRX reconnection experiment (Yamada 1999; Yamada et al. 2010).

3. MAGNETOHYDRODYNAMIC TURBULENCE

Magnetized turbulence is a tough and complex problem (see Biskamp (2003) and references therein). A broad outlook on the astrophysical implications of the turbulence can be found in a review by Elmegreen & Scalo (2004), while the effects of turbulence on molecular clouds and star formation are reviewed in McKee & Ostriker (2007) and Ballesteros-Paredes et al. (2007). However, the issues of turbulence spectrum and its anisotropies, we feel, are frequently given less attention than they deserve.

In spite of its complexity, the turbulent cascade is self-similar and universal (see Monin & Yaglom (1975)) over its inertial range. The physical variables are proportional to simple powers of the eddy sizes over a large range of sizes, leading to scaling laws expressing the dependence of certain non-dimensional combinations of physical variables on the eddy size. Robust scaling relations can predict turbulent properties on the whole range of scales, including those that no large-scale numerical simulation can hope to resolve. These scaling relations are extremely important for obtaining an insight into processes on the small scales. In the interstellar medium, the inertial range of fluctuations is very appreciable, i.e. from hundreds of kilometers to dozens of parsecs (see Armstrong et al. (1995) for the data at small scale, and Chepurnov & Lazarian (2010) for its extension to pc scales) and therefore the details of the large-scale driving and of the small-scale dissipation become unimportant for the properties of turbulence at the long range of intermediate scales.

The presence of a magnetic field makes MHD turbulence anisotropic (Montgomery & Turner (1981); Matthaeus et al. (1983); Shebalin et al. (1983); Higdon (1984); Goldreich & Sridhar (1995), see Oughton et al. (2003) for a review). The relative importance of hydrodynamic and magnetic forces changes with scale, so the anisotropy of MHD turbulence does too. If the turbulence spectrum and its anisotropy are known, then many astrophysical results, e.g. the dynamics of dust, scattering and acceleration of energetic particles, thermal conduction, can be obtained.

3.1. *The Goldreich-Sridhar picture of strong and weak MHD turbulence*

The standard theory for Alfvénic turbulence is currently the one suggested by Goldreich & Sridhar (1995) [henceforth GS95]. The cornerstone of the theory is the so-called critical balance, which is the rough equality between the timescale of the eddy-turnovers perpendicular to the local magnetic field and the periods of waves associated with the eddies³. The predictions of the GS95 model are in rough agreement with numerical simulations (Cho & Vishniac 2000; Maron & Goldreich 2001; Cho et

al. 2002; Beresnyak & Lazarian 2006), although some disagreement in terms of the measured spectral slope was noted. This disagreement produced a flow of papers with suggestions to improve the GS95 model by including additional effects like dynamical alignment (Boldyrev 2005, 2006), polarization intermittency (Beresnyak & Lazarian 2006), non-locality (Gogoberidze 2007). More recent studies in Beresnyak & Lazarian (2009, 2010) and Beresnyak (2011) indicate that all numerical simulations performed to date may not have sufficiently extended inertial range to get the actual spectral slope and therefore worries about the “inconsistency” of the GS95 model may be premature.

We shall add parenthetically that in a number of applications the empirical so-called composite 2D/slab model of magnetic fluctuations is used (see Bieber et al. (1994)). In the latter model, which is also known as *two-component model*, it is assumed that interplanetary fluctuations can be described as a superposition of fluctuations with wave vectors parallel to the ambient large-scale magnetic field (so-called *slab modes*) and perpendicular to the mean field (so-called *two-dimensional modes*). This model results in a *Maltese cross* structure of magnetic correlations. This model was developed to account for the solar wind observations, which it does well by adjusting the intensity of two components (see, e.g., Matthaeus et al. (1990)). From the theoretical point of view, as well as from the point of numerical testing, this standard model is rather vulnerable. Indeed, 2D fluctuations are consistent with the theory of weak Alfvénic turbulence (see Ng & Bhattacharjee (1996); Lazarian & Vishniac (1999); Galtier et al. (2000)), but this analytical theory predicts strengthening of the cascade with decrease of the turbulence scale. Therefore 2D fluctuations can describe Alfvénic turbulence only over a limited range of scales. In addition, slab modes do not arise naturally in MHD numerical simulations with large-scale driving (see Cho & Lazarian (2002, 2003)). We do not discuss this model of MHD turbulence further, as it does not have physical motivation and is not supported by numerical modeling. It may be treated as a parameterization of a particular type of magnetic perturbation dominated by the peculiarities of driving. On the contrary, we consider turbulence which has an extended universal inertial range over which the effects of driving and dissipation are negligible.

Below we shall adopt the GS95 model of turbulence to describe the Alfvénic part of MHD turbulent fluctuations. Note that the Alfvénic perturbations are most important for magnetic field wandering which, according to LV99, is the process that enables fast reconnection. The GS95 model can be generalized to compressible turbulence and even for supersonic motions numerical calculations show that the Alfvénic perturbations exhibit GS95 scaling (Cho & Lazarian 2002, 2003; Kowal & Lazarian 2010). In what follows we consider MHD turbulence where the flows of energy in the opposite directions are balanced. When this is not true, i.e. when the turbulence has non-zero cross-helicity, the properties of turbulence depart substantially from the GS95 model⁴.

³ This equality is frequently expressed following GS95 convention in terms of parallel and perpendicular wavenumbers. However, as the wavenumbers are defined in the global magnetic field frame of reference, this may be misleading. Critical balance is a condition satisfied only in the frame related to the local magnetic field. The original GS95 work also uses closure relations which are true only in the global frame of reference. The importance of the local system of reference was implicit in Kraichnan (1965) but was first discussed within the GS95 model in subsequent publications (LV99, Cho & Vishniac (2000); Maron & Goldreich (2001); Cho et al. (2002); Cho & Lazarian (2002)).

⁴ Among the existing theories of imbalanced turbulence (see Lithwick et al. (2007); Beresnyak & Lazarian (2008); Chandran (2008); Perez & Boldyrev (2009), all, but Beresnyak & Lazarian

Similarly, we shall not discuss MHD turbulence at high magnetic Prandtl numbers, when the field-parallel viscosity is much larger than resistivity (see Cho et al. (2002, 2003); Lazarian et al. (2004)). This viscosity-dominated regime has important consequences for turbulence in partially ionized gas (Lazarian et al. 2004), while our present study deals mostly with conducting fluids and fully ionized plasmas. Our considerations do apply for large magnetic Prandtl numbers at scales greater than the viscous length, if the kinetic Reynolds number is also large.

The nature of Alfvénic cascade is expressed through the critical balance condition in the GS95 model of strong turbulence, namely,

$$\ell_{\parallel}^{-1} v_A \sim \ell_{\perp}^{-1} \delta u_{\ell}, \quad (2)$$

where δu_{ℓ} is the eddy velocity, while ℓ_{\parallel} and ℓ_{\perp} are, respectively, eddy scales parallel and perpendicular to the local direction of magnetic field. The critical balance condition states that the parallel size of an eddy is determined by the distance Alfvénic perturbation can propagate during an eddy turnover. The qualifier “local” is important⁵, as no universal relations exist if eddies are treated in respect to the global mean magnetic field (LV99, Cho & Vishniac (2000); Maron & Goldreich (2001); Lithwick & Goldreich (2001); Cho et al. (2002)). Combining (2) with the Kolmogorov cascade notion, i.e. that the energy transfer rate is $\delta u_{\ell}^2/(\ell_{\perp}/\delta u_{\ell}) = \text{const}$ one gets $\delta u_{\ell} \sim \ell_{\perp}^{1/3}$ and $\ell_{\parallel} \sim \ell_{\perp}^{2/3}$. If the turbulence injection scale is L_i , then $\ell_{\parallel} \propto L_i^{1/3} \ell_{\perp}^{2/3}$, which shows that the eddies get very anisotropic for small ℓ_{\perp} . Recent measurements of anisotropy in the solar wind are consistent with these predictions (Podesta 2010; Wicks et al. 2010, 2011).

Critical balance is a feature of strong turbulence, which is the case when the turbulence is injected isotropically with velocity amplitude $u_L = v_A$. If the turbulence is injected at velocities $u_L \ll v_A$ (or anisotropically with $L_{i,\parallel} \ll L_{i,\perp}$), then the cascade is weak, with ℓ_{\perp} of the eddies decreasing but $\ell_{\parallel} = L_i$ unchanged by resonant wave interactions (Montgomery & Matthaeus 1995; Lazarian & Vishniac 1999; Galtier et al. 2000). In other words, as a result of the weak cascade the eddies get thinner, but preserve the same length along the local magnetic field. The energy cascade rate in the weak MHD turbulence regime is (Kraichnan 1965; Ng & Bhattacharjee 1996; Lazarian & Vishniac 1999)

$$\varepsilon \approx \tau_{corr} \delta u_{\ell}^4 / \ell_{\perp}^2 \approx \delta u_{\ell}^4 L_i / v_A \ell_{\perp}^2, \quad (3)$$

where $\tau_{corr} = L_i/v_A$ is the decorrelation time due to large-scale Alfvén waves. This implies that the scaling of weak turbulence is

$$\delta u_{\ell} \sim u_L (\ell_{\perp}/L_i)^{1/2}. \quad (4)$$

(2008) seem to contradict to numerical testing in Beresnyak & Lazarian (2009, 2010). Solar wind presents a system with imbalanced turbulence. In compressible media the imbalance decreases due to reflecting of waves from pre-existing density fluctuations and due to the development of parametric instabilities.

⁵ To stress the difference between local and global systems here we do not use the language of k -vectors. Wavevectors parallel and perpendicular to magnetic fields can be used, if only the wavevectors are understood in terms of a wavelet transform defined with the local reference system rather than ordinary Fourier transform defined with the mean field system.

For weak turbulence initially the LHS of Eq. (2) is larger than the RHS, but as ℓ_{\perp} decreases this eventually makes Eq. (2) satisfied.

Comparing Eqs. (4) and (2) for $\ell_{\parallel} = L_i$, one can see that the transition to the strong MHD turbulence happens at the scale $L_{trans} = L_i M_A^2$ and the velocity at this scale is $v_{trans} = u_L M_A$, with $M_A = u_L/v_A \ll 1$ the Alfvénic Mach number of the turbulence (Lazarian & Vishniac 1999; Lazarian 2006). Thus, weak turbulence has a limited, i.e. $[L_i, L_i M_A^2]$ inertial interval before it gets into the regime of strong turbulence at smaller scales. Note that weak and strong are not the characteristics of the amplitude of turbulent perturbations, but the strength of non-linear interactions (see more discussion in Cho et al. (2003)) and very weak Alfvénic perturbations can correspond to a strong Alfvénic cascade.

While GS95 assumed that the turbulent energy is isotropically injected with amplitude $u_L = v_A$ at the scale L_i , LV99 provided general relations for the turbulent scaling at small scales for the case that the injection velocity u_L is less or equal to v_A . Combining Eqs. (2), (3) at $\ell_{\perp} = L_i$, and the constant flux condition, we get the relations between the parallel and perpendicular scales of eddies in the strong GS95 cascade range, which can be written in terms of ℓ_{\parallel} and ℓ_{\perp} (LV99):

$$\ell_{\parallel} \approx L_i \left(\frac{\ell_{\perp}}{L_i} \right)^{2/3} M_A^{-4/3} \quad (5)$$

$$\delta u_{\ell} \approx u_L \left(\frac{\ell_{\perp}}{L_i} \right)^{1/3} M_A^{1/3}. \quad (6)$$

As we discuss later, the present day debates of whether GS95 approach should be augmented by additional concepts like “dynamical alignment”, “polarization”, “non-locality” (Boldyrev 2006; Beresnyak & Lazarian 2006, 2009; Gogoberidze 2007) do not change the nature of the reconnection of the weakly turbulent magnetic field.

The above relations were exploited in LV99 to estimate magnetic field lateral diffusion. Moving a distance s along field-lines, one finds that a pair of lines initially a distance $\ell_{\perp}^{(0)}$ apart at $s = 0$ separate at the rate

$$\frac{d}{ds} \ell_{\perp} \simeq \frac{\delta b_{\ell}}{B_0} \simeq \frac{\delta u_{\ell}}{v_A}. \quad (7)$$

Substituting the scaling given by Eq. (6), one can solve to obtain

$$\ell_{\perp}^2 \simeq (s^3/L_i) M_A^4 \quad (8)$$

when $L_i > s \gg \ell_{\perp}^{(0)}$. This result applies only until $s = L_i$ when $\ell_{\perp}(s) \simeq L_{trans}$, the transitional scale between weak and strong turbulence. Since $\ell_{\parallel} = L_i$ is constant in weak turbulence, there are no eddies with parallel length-scale greater than L_i . Thus, as s continues to increase, the lines encounter at every increment of s by L_i a new turbulent eddy and undergo a further separation by L_{trans} . The result is a diffusive random-walk with

$$\ell_{\perp}^2 \simeq L_{trans}^2 (s/L_i) \simeq s L_i M_A^4 \quad (9)$$

for $s > L_i$. The results (8),(9) are exact analogues for magnetic field-lines of the 2-particle diffusion of Lagrangian particle trajectories discussed in the next section and can be obtained more rigorously by the methods

employed there. Since this discussion is somewhat out of logical order, we defer it to Appendix A.

A remarkable feature of (8),(9) is their complete lack of dependence on the initial separation $\ell_{\perp}^{(0)}$ of the lines. As we shall see below, it is this property which leads in the LV99 model to fast magnetic reconnection in MHD turbulence independent of the microscopic resistivity. The shear-Alfvénic component of turbulence is the most important for magnetic field-line wandering. Cho & Lazarian (2002, 2003) demonstrated the correspondence of the Alfvénic component of compressible MHD turbulence, obtained by decomposition into fundamental modes, to the Alfvénic turbulence in incompressible media. Thus we shall use the relations above to treat magnetic fields in realistically compressible astrophysical fluids.

3.2. Lagrangian particle dynamics in MHD turbulence

It will be important for what follows to develop a theoretical understanding of the dynamics of Lagrangian particles in MHD turbulence. Here one means by “particles” not the microscopic charged constituents of the plasma (electrons and ions), but instead macroscopically small and microscopically large parcels of plasma fluid. The Lagrangian perspective is particularly important to understand turbulent processes and there has been an explosion of research in this area over the last decade (see Falkovich et al. (2001); Salazar & Collins (2009); Toschi & Bodenschatz (2009)). There has been relatively little work done on the Lagrangian properties of MHD turbulence, however, where the effects of Alfvén waves on fluid particle motion can be significant. We shall here develop a theory of Lagrangian MHD turbulence at the level of GS95 phenomenology by a semi-quantitative use of the equations of motion.

We begin with the simplest case of 1-particle or Taylor diffusion, which concerns the displacement $\delta\mathbf{x}(t) = \mathbf{x}(t) - \mathbf{x}_0$ of a Lagrangian fluid particle from its initial position as it moves according to the advection equation

$$\frac{d}{dt}\mathbf{x}(t) = \mathbf{U}(t) \quad (10)$$

with $\mathbf{U}(t) = \mathbf{u}(\mathbf{x}(t), t)$ the Lagrangian particle velocity. At early times $t \ll \tau_L$ or short compared with the eddy-turnover time $\tau_L = u_L/L_i$, motion is ballistic with $\langle x^2(t) \rangle \sim u_L^2 t^2$. It was shown by Taylor (1921) that the motion is diffusive $\langle \delta x_i(t) \delta x_j(t) \rangle \sim 2D_{ij}t$ at times $t \gg \tau_L$, with the eddy-diffusivity

$$D_{ij} = \frac{1}{2} \int_{-\infty}^{+\infty} dt \langle U_i(t) U_j(0) \rangle. \quad (11)$$

These results carry over unchanged to MHD turbulence. For the weak magnetic field case with $M_A \geq 1$ (including hydrodynamic turbulence with $M_A = \infty$), $\langle U(t)U(0) \rangle \sim u_L^2 e^{-t/\tau_L}$ (Mordant et al. 2001; Busse & Müller 2008). Thus,

$$D \sim u_L^2 \tau_L \sim u_L L_i, \quad (12)$$

exactly as for hydrodynamics of neutral fluids. When $M_A \simeq 1$, one expects slight differences from the hydrodynamic case due to anisotropy of MHD turbulence. For example, the velocity components U_{\parallel} and U_{\perp} ought to have times scales $\tau_{L\parallel} > \tau_{L\perp}$, since the parallel components are pseudo-Alfvénic modes with essentially passive

scalar dynamics, due to reduced nonlinear interactions. Thus, one expects that $D_{\parallel} > D_{\perp}$, as indeed observed in simulations of Busse & Müller (2008). For $M_A \ll 1$, there are more profound effects due to the strong field. Weak MHD turbulence becomes relevant, for which the nonlinear time-scale is $\tau_L = M_A^{-2} \omega_A^{-1}$ with $\omega_A = v_A/L_i$ the forcing-scale Alfvén wave frequency. (See Kraichnan (1965); Sridhar & Goldreich (1994); Galtier et al. (2000); cf. also eq.(25) below for $\ell_{\perp} = L_i$). Furthermore, fluid particles in the Lagrangian frame experience fast shear-Alfvénic oscillations in velocity, so that

$$\langle U_{\perp}(t) U_{\perp}(0) \rangle \sim u_L^2 \text{Re} \left(e^{i\omega_A t - t/\tau_L} \right). \quad (13)$$

Thus, (11) gives

$$D_{\perp} \sim u_L^2 \frac{\tau_L}{(\omega_A \tau_L)^2 + 1}. \quad (14)$$

Since $\omega_A \tau_L = M_A^{-2} \gg 1$,

$$D_{\perp} \sim u_L^2 \frac{1}{\omega_A^2 \tau_L} \sim u_L L M_A^3. \quad (15)$$

There is a large reduction in the turbulent 1-particle diffusivity due to inefficient advection by Alfvén wave modes. Physically, the velocity of a particle transported by Alfvén wave turbulence experiences rapid oscillations in sign that lead to large cancellations in the net displacement.

We note as an aside that the Taylor 1-particle diffusivity given by (11) coincides with the effective turbulent diffusivity of an advected scalar field (e.g. temperature), whenever the “eddy-diffusivity” concept is valid. In order for such a description of the turbulence effects to be accurate, the scalar fields must vary slowly on spatial scales of order L_i and, in that case, they experience an augmentation of their molecular diffusivity by a “turbulent diffusivity” precisely given by (11). Our results (12),(15) thus reproduce those of Lazarian (2006) for the eddy thermal diffusivity in MHD turbulence. Of course, in general, when there is no separation between the scale of variation of the scalar field and the scale L_i of the turbulence, then the gradient-transport assumption breaks down (e.g. Tennekes & Lumley (1972), §2.3). In such cases, eddy-diffusivity models can produce erroneous results and may suffice only for order-of-magnitude estimates of scalar transport.

Two-particle turbulent diffusion or Richardson diffusion concerns instead the separation $\ell(t) = \mathbf{x}(t) - \mathbf{x}'(t)$ between a pair of Lagrangian fluid particles. It was proposed by Richardson (1926) that this separation grows in turbulent flow according to the formula

$$\frac{d}{dt} \langle \ell_i(t) \ell_j(t) \rangle = \langle D_{ij}(\ell) \rangle \quad (16)$$

with a scale-dependent eddy-diffusivity $D(\ell)$. In hydrodynamic turbulence Richardson deduced that $D(\ell) \sim \varepsilon^{1/3} \ell^{4/3}$ (see also Obukhov (1941)) and thus $\ell^2(t) \sim \varepsilon t^3$. An analytical formula for the 2-particle eddy-diffusivity was later derived by Batchelor (1950); Kraichnan (1966):

$$D_{ij}(\ell) = \int_{-\infty}^0 dt \langle \delta U_i(\ell, 0) \delta U_j(\ell, t) \rangle \quad (17)$$

with $\delta U_i(\ell, t) \equiv U_i(\mathbf{x} + \ell, t) - U_i(\mathbf{x}, t)$ the relative velocity at time t of a pair of fluid particles which were at positions \mathbf{x} and $\mathbf{x} + \ell$ at time 0. This formula is the analogue for 2-particle eddy-diffusivity of Taylor's formula (11) for 1-particle diffusivity and it is valid in MHD turbulence just as in hydrodynamic turbulence. We can assume similarly as before that

$$\langle \delta U_\perp(\ell, 0) \delta U_\perp(\ell, t) \rangle \sim \delta u_\ell^2 \text{Re} \left(e^{i\omega_A \ell t - |t|/\tau_\ell} \right), \quad (18)$$

where $\omega_{A,\ell}$ is the Alfvén wave-frequency and τ_ℓ is the nonlinear interaction time both at length-scale ℓ . The result for the 2-particle diffusivity analogous to (14) is

$$D_\perp(\ell) \sim \delta u_\ell^2 \frac{\tau_\ell}{(\omega_A \ell \tau_\ell)^2 + 1}. \quad (19)$$

We consider here only the case $M_A < 1$, for which weak turbulence holds when $\ell_\perp > L_{trans}$ and strong turbulence when $\ell_\perp < L_{trans}$. We shall also treat particle dispersion only in the direction perpendicular to the background magnetic field, since this is what is required later for the application to reconnection. See Busse & Müller (2008); Eyink (2011) for some discussion of particle dispersion in the field-parallel direction.

We first consider the strong turbulence regime when $\ell_\perp(t) < L_{trans}$. In this case, $\tau_\ell = \varepsilon^{-1/3} \ell_\perp^{2/3}$ and $\omega_A \ell = v_A / \ell_\parallel$, where ℓ_\parallel is the parallel scale of the turbulent eddy with perpendicular scale $\ell_\perp(t)$ (and not the particle-separation $\ell_\parallel(t)$ in the field-parallel direction). Because of the critical balance condition in strong MHD turbulence,

$$\omega_A \ell \tau_\ell = (\text{const.}) \quad (20)$$

In that case

$$\frac{\tau_\ell}{(\omega_A \ell \tau_\ell)^2 + 1} = f \tau_\ell \quad (21)$$

with $f = [(\omega_A \ell \tau_\ell)^2 + 1]^{-1} < 1$ a constant factor which represents the reduced efficiency of particle transport due to wave oscillations. Eq.(19) gives

$$D_\perp(\ell) \sim f \delta u_\ell^2 \tau_\ell \sim f \varepsilon^{1/3} \ell_\perp^{4/3} \quad (22)$$

with the same scaling as the original Richardson (1926) 2-particle diffusivity. Thus,

$$\ell_\perp^2(t) \sim f^3 \varepsilon t^3, \quad (23)$$

showing the same t^3 -law as in the hydrodynamic case, with only a reduced coefficient.

We now consider the weak turbulence regime with $\ell_\perp(t) > L_{trans}$. The pair of particles are advected apart by a turbulent eddy with corresponding perpendicular length-scale but with $\ell_\parallel = L_i$ for all eddies, so that $\omega_A \ell = \omega_A = v_A / L_i$, independent of ℓ_\perp . The nonlinear interaction time τ_ℓ is estimated from

$$\frac{\delta u_\ell^2}{\tau_\ell} = \varepsilon = \frac{u_L^4}{v_A L_i} \quad (24)$$

together with (4) to give

$$\tau_\ell = (\ell_\perp / v_A) M_A^{-2}. \quad (25)$$

Since

$$\omega_A \tau_\ell = \ell_\perp / L_{trans} \gg 1, \quad (26)$$

it follows from (19) that

$$D_\perp(\ell) \sim \frac{\delta u_\ell^2}{\omega_A^2 \tau_\ell} \sim \frac{\varepsilon}{\omega_A^2} = u_L L_i M_A^3. \quad (27)$$

Note that $D_\perp(\ell)$ is identical with the 1-particle diffusivity D_\perp given by (15) and, in particular, is independent of ℓ_\perp . It thus follows that in the weak turbulence regime

$$\ell_\perp^2(t) \sim u_L L_i M_A^3 t \quad (28)$$

and 2-particle diffusion is identical to 1-particle diffusion. It should be possible to verify this interesting prediction by analytical methods of weak turbulence theory for Lagrangian particle motion (see Balk (2002)). The crossover between the dispersion laws (23) and (28) occurs at the time $t_c \sim L_i / v_A = M_A t_L$.

4. FLUX-FREEZING AND SPONTANEOUS STOCHASTICITY

There is an intimate connection between magnetic reconnection and flux-freezing. Indeed, some authors consider that reconnection is precisely any topology-changing evolution of the magnetic field due to nonexistence of a smooth, flux-preserving velocity field. For example, Greene (1993) proposed to define reconnection “as evolution in which it is not possible to preserve the global identification of some field lines.” See also Hornig & Schindler (1996). For this reason, a necessary prelude to our treatment of turbulent magnetic reconnection is a careful discussion of how flux-freezing operates in MHD turbulence.

4.1. Stochastic flux-freezing

Magnetic field lines do not move. Magnetic fields evolve under the Maxwell equations, of course, but single field lines do not possess an identity over time. As has long been understood, magnetic field-line motion is a “metaphysical” concept, defined by convention and not testable by experiment (cf. Newcomb (1958); Vasyliunas (1972); Alfvén (1976)). Any line-velocity which leads to agreement with the Maxwell equations for the magnetic field evolution is equally acceptable, e.g. the bulk plasma velocity \mathbf{u} in ideal MHD, the $\mathbf{E} \times \mathbf{B}$ drift velocity in low energy plasmas, etc. Flux-freezing is such a powerful heuristic tool, however, that it is often forgotten that the concept of magnetic line-motion rests on certain conventions and is not a direct, physical reality. In particular, real physical plasmas always possess a Spitzer plasma resistivity (along with other possible forms of non-ideality, such as electron inertia, electron pressure anisotropy, etc.) The validity of flux-freezing in the standard sense depends upon careful consideration of the limit when such non-ideal effects are taken to be small.

Since field-lines do not move in reality, one is free to adopt any convention for their motion which is consistent with the dynamical equations. Let us for the moment assume the validity of the resistive induction equation

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \lambda \Delta \mathbf{B} \quad (29)$$

with $\lambda = \eta c^2 / 4\pi$ the magnetic diffusivity for a simple scalar resistivity η .⁶ A smooth, deterministic velocity

⁶ It may be that the flux-freezing constraint is broken in actual

field \mathbf{u}_* is flux-preserving if the magnetic field obeys

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{u}_* \times \mathbf{B}). \quad (30)$$

Unfortunately, for the resistive MHD model in three space dimensions, there is generally no flux-preserving velocity whatsoever. For example, Wilmot-Smith et al. (2005) provide a counterexample with a closed magnetic field-line through which the flux is changing in time, so that no deterministic flux-preserving velocity can exist. This seems to leave one without any consistent and meaningful approach to discuss field-line “motion” in the presence of Ohmic resistivity.

On the other hand, magnetic line-motion in a resistive MHD plasma may be very naturally regarded as *stochastic* (Eyink 2009). Let the plasma fluid velocity be $\mathbf{u}(\mathbf{x}, t)$, assumed here to be divergence-free for an incompressible plasma⁷. Then we may define stochastic Lagrangian trajectories $\tilde{\mathbf{x}}(\mathbf{a}, t)$ by the initial-value problem

$$\begin{cases} \frac{d\tilde{\mathbf{x}}}{dt}(\mathbf{a}, t) = \mathbf{u}(\tilde{\mathbf{x}}(\mathbf{a}, t), t) + \sqrt{2\lambda}\tilde{\boldsymbol{\eta}}(t), \\ \tilde{\mathbf{x}}(\mathbf{a}, t_0) = \mathbf{a}, \end{cases} \quad (31)$$

where $\tilde{\boldsymbol{\eta}}(t)$ is a three-dimensional vector white-noise process, delta-correlated in time. For any initial magnetic field $\mathbf{B}_0(\mathbf{x})$ one can form the random field

$$\tilde{\mathbf{B}}(\mathbf{x}, t) = \mathbf{B}_0(\mathbf{a}) \cdot \nabla_{\mathbf{a}} \tilde{\mathbf{x}}(\mathbf{a}, t)|_{\tilde{\mathbf{a}}(\mathbf{x}, t)} \quad (32)$$

where $\tilde{\mathbf{a}}(\mathbf{x}, t)$ is the spatial inverse map to $\tilde{\mathbf{x}}(\mathbf{a}, t)$. If the flow map were deterministic, then (32) would be the standard *Lundquist formula* for the magnetic field. What is true in the present case with $\lambda > 0$ is that the ensemble-average field

$$\mathbf{B}(\mathbf{x}, t) = \overline{\tilde{\mathbf{B}}(\mathbf{x}, t)} \quad (33)$$

over all independent realizations of the white-noise is the exact solution of the resistive induction equation (29) with initial condition

$$\mathbf{B}(\mathbf{x}, t_0) = \mathbf{B}_0(\mathbf{x}). \quad (34)$$

In fact, (31)-(33) are mathematically equivalent to (29) and (34). It should not be too surprising that averaging over Brownian motions will reproduce the Laplacian diffusion term $\lambda \Delta \mathbf{B}$.

The eqs.(31)-(33) are equivalent to the following path-integral formula, derived in Eyink (2011):

$$\begin{aligned} \mathbf{B}(\mathbf{x}, t) &= \int_{\mathbf{a}(t)=\mathbf{x}} \mathcal{D}\mathbf{a} \, \mathbf{B}_0(\mathbf{a}(t_0)) \cdot \mathcal{J}(\mathbf{a}, t) \\ &\times \exp \left(-\frac{1}{4\lambda} \int_{t_0}^t d\tau |\dot{\mathbf{a}}(\tau) - \mathbf{u}^\nu(\mathbf{a}(\tau), \tau)|^2 \right) \end{aligned} \quad (35)$$

where \mathbf{B} is interpreted as a 3-dimensional row vector and $\mathcal{J}(\mathbf{a}, \tau)$ is a 3×3 matrix satisfying the following ODE

astrophysical plasmas by some other microscopic mechanism (e.g. electron inertia, finite electron gyroradius, etc.). We shall consider such alternatives below (section 4.3) and show that they do not alter our final conclusions. In particular, we argue that the large-scale global reconnection rate in a turbulent plasma is independent of whatever microscopic plasma process produces field-line breaking. It is thus pedagogically useful to begin with the simplest case.

⁷ The assumption of incompressibility is convenient to simplify the arguments and, also, realistic when considering the RMHD dynamics of shear-Alfvén modes. As a matter of fact, the formulas (31)-(37) below all generalize to the compressible case. See Eyink (2011).

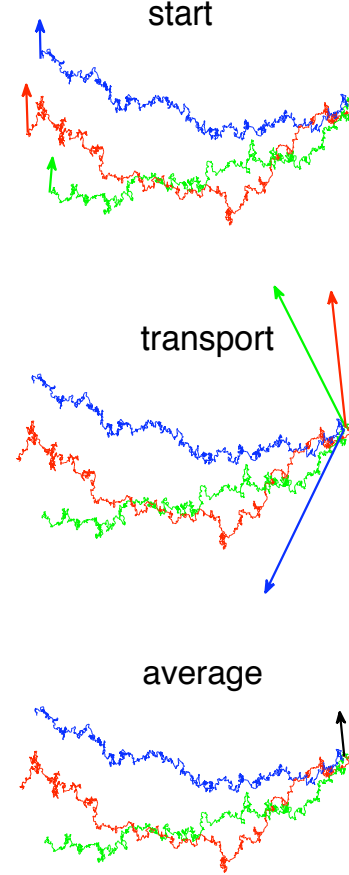


FIG. 1.— *Illustration of the stochastic Lundquist formula.* Three stochastic Lagrangian trajectories running backward in time from a common point are shown in red, blue, green. Starting field vectors, represented by correspondingly colored arrows, are transported along the trajectories, stretched and rotated, to the common final point. These “virtual field vectors” are then averaged to give the resultant magnetic field at that point, indicated by the black arrow.

along the trajectory $\mathbf{a}(\tau)$:

$$\begin{cases} \frac{d}{d\tau} \mathcal{J}(\mathbf{a}, \tau) = \mathcal{J}(\mathbf{a}, \tau) \nabla_{\mathbf{x}} \mathbf{u}(\mathbf{a}(\tau), \tau) \\ \mathcal{J}(\mathbf{a}, t_0) = \mathbf{I}. \end{cases} \quad (36)$$

By applying $\nabla_{\mathbf{a}}$ to (31) it is easy to see that $\mathcal{J}(\mathbf{a}, \tau) = \nabla_{\mathbf{a}} \tilde{\mathbf{x}}(\mathbf{a}(t_0), \tau)$ and thus we may identify

$$\tilde{\mathbf{B}}(\mathbf{a}(\tau), \tau) = \mathbf{B}_0(\mathbf{a}(t_0)) \cdot \mathcal{J}(\mathbf{a}, \tau). \quad (37)$$

As often with such path-integration techniques, only the initial field $\mathbf{B}_0(\mathbf{x})$ and the final field $\mathbf{B}(\mathbf{x}, t)$ have real physical meaning and the intermediate random fields $\tilde{\mathbf{B}}(\mathbf{x}, t)$ are merely calculational devices. We shall refer to the latter as the “virtual” magnetic fields, since they enter as non-real, intermediate states between two physical fields at different times. Equation (33) [or (35)] shows that summing over all of the “virtual” fields at any instant reproduces the true, physical magnetic field. It follows from (32) [or (37)] that each “virtual” field $\tilde{\mathbf{B}}$ is topologically equivalent to the initial field \mathbf{B}_0 and is simply stretched, deformed and advected by its own

stochastic flow $\tilde{\mathbf{x}}$. Field-lines for two different “virtual” fields that arise from statistically independent flows may pass directly through each other as they move, since they correspond to completely different random realizations. The topology of the field-lines is only changed by the final averaging step, which corresponds to resistive gluing of all the “virtual” magnetic field vectors which arrive to the same point \mathbf{x} at time t . It is this resistive averaging which reconnects the lines of the initial magnetic field \mathbf{B}_0 . See Fig. 1 for a pictorial representation of this process.

We shall refer to either the relations (31)-(33) or the equivalent path-integral formula (35)-(36) as the *stochastic flux-freezing* (SFF) relations. They provide a consistent approach to describe the “motion” of magnetic field-lines in the presence of resistivity. Setting $\lambda = 0$ one formally recovers the standard deterministic flux-freezing relations of ideal MHD, but this conclusion is not rigorous. A careful discussion is required of the limiting process involved.

4.2. Spontaneous stochasticity

Standard flux-freezing arguments are invoked in the limit of high conductivity or vanishingly small magnetic diffusivity. It is usually assumed that flux-freezing in the conventional sense will hold better and better as $\lambda \rightarrow 0$. This is not true, however, when the plasma flow is turbulent. Instead, flux-freezing in this circumstance breaks down in a completely novel and unexpected way: it remains stochastic (Eyink 2007, 2011).

Consider first the simpler situation of a smooth, laminar solution of the resistive MHD equations in the limit of very small λ . Define the standard (deterministic) Lagrangian flow map by

$$\begin{cases} \frac{d\mathbf{x}}{dt}(\mathbf{a}, t) = \mathbf{u}(\mathbf{x}(\mathbf{a}, t), t), \\ \mathbf{x}(\mathbf{a}, 0) = \mathbf{a}. \end{cases} \quad (38)$$

It is not hard to see under the stated assumptions that⁸

$$|\tilde{\mathbf{x}}(\mathbf{a}, t) - \mathbf{x}(\mathbf{a}, t)| = O(\sqrt{\lambda t}) \quad (39)$$

for typical realizations of the “virtual” flows $\tilde{\mathbf{x}}$. The real magnetic field is also given to very good accuracy by the standard Lundquist formula, with errors that vanish as $\lambda \rightarrow 0$. Thus, each “virtual” field line is wiggling a small distance $\sim \sqrt{\lambda t}$ around a particular physical field-line for very small λ (and not too large times t). For this reason, one does not need to make use of the concept of “virtual” field lines for smooth, laminar solutions of the MHD equations. The lines of force of the real, physical magnetic field are themselves “nearly” frozen-in for very small resistivities and one does not need to make an essential distinction between real and “virtual” fields.

This is not the case for rough, turbulent solutions of the resistive MHD equations, even for vanishingly small λ . By “rough” fields we mean here more precisely that \mathbf{u}, \mathbf{B}

which solve the MHD equations have power-law energy spectra $\propto k^{-n}$, with $1 < n < 3$, for a long range of wavenumbers k with kinematic viscosity ν and magnetic diffusivity λ both small. For example, in the GS95 theory of MHD turbulence the three-dimensional (anisotropic) energy spectra for both \mathbf{u}, \mathbf{B} can be written in the form

$$E(k_\perp, k_\parallel) \sim v_A^2 M_A^{4/3} L_i^{-1/3} k_\perp^{-10/3} f\left(\frac{k_\parallel L_i^{1/3}}{k_\perp^{2/3} M_A^{4/3}}\right). \quad (40)$$

where we added, compared to the original expressions, the dependence on M_A following LV99. The original GS95 theory was formulated for $M_A \equiv 1$, as we mentioned earlier. Note, that the above expression is expected to be valid for large enough k_\parallel and k_\perp . The function f is not specified in GS95, but Cho, Lazarian & Vishniac (2002) showed that it can be fitted by a Castaing function (see Castaing, Gagne & Hopfinger 1990), which, in its turn, outside the vicinity of $k_\parallel = 0$, can be approximated by an exponent. As we mentioned earlier, k_\parallel and k_\perp in the expression are not the wavevectors in the conventional sense as the corresponding scales should be measured in the local system of reference⁹

One thus finds that $E(k_\perp) \propto k_\perp^{-5/3}$ and $E(k_\parallel) \propto k_\parallel^{-2}$, so that the fields predicted by GS95 are “rough” in our sense. In this case, (39) is invalid except for very short times, due to the properties of 2-particle Richardson diffusion. For example, in the transverse direction perpendicular to the local magnetic field, (39) holds only for times less than $t_\lambda = O(\sqrt{\lambda/\varepsilon})$. If one considers a pair of independent “virtual” flows $\tilde{\mathbf{x}}, \tilde{\mathbf{x}}'$, then for times $t \gg t_\lambda$ in the root-mean-square sense

$$|\tilde{\mathbf{x}}_\perp(\mathbf{a}, t) - \tilde{\mathbf{x}}'_\perp(\mathbf{a}, t)| \sim (\varepsilon t^3)^{1/2}. \quad (41)$$

See eq.(23). This result is completely independent of magnetic diffusivity, with very remarkable consequences. As $\lambda, \nu \rightarrow 0$, and thus also $t_\lambda \rightarrow 0$, the two realizations stay far apart for *all* times t and do not collapse to a single real particle trajectory. This phenomenon has been called *spontaneous stochasticity* because the distance between “virtual” particle trajectories stays finite (and random) in the limit, similar to the way that a spontaneous magnetization develops in a ferromagnet below the Curie temperature when an external magnetic field is taken to zero (Bernard et al. 1998; Gawędzki & Vergassola 2000; E & vanden Eijnden 2000, 2001; Chaves et al. 2003; Kupiainen 2003). In the limit $\lambda, \nu \rightarrow 0$ all of the “virtual” flows $\tilde{\mathbf{x}}(\mathbf{a}, t)$ solve the deterministic initial-value problem (38), which then possesses infinitely many solutions! For more discussion of spontaneous stochasticity and for a review of experimental and numerical evidence of this remarkable phenomenon, see Eyink (2011).

It is an immediate consequence that in a “rough” turbulent velocity field, the conventional notion of flux-freezing must break down. The violations are not small. Taking the square of (39) as the conventional estimate for line-slippage, $\langle \ell_\perp^2 \rangle_{\text{frozen}} \sim \lambda t$, and comparing with

⁸ A more precise statement is that $\langle |\tilde{\mathbf{x}}(\mathbf{a}, t) - \mathbf{x}(\mathbf{a}, t)|^2 \rangle \leq \frac{6\lambda}{K}(e^{Kt} - 1)$ where K is the “Lipschitz constant” of the velocity field \mathbf{u} . See Freidlin & Wentzell (1984), Ch.2. In the presence of Lagrangian chaos this upper bound may be attained, with K the leading Lyapunov exponent. Note that for $t \ll 1/K$ the upper bound reduces to $6\lambda t$, the usual diffusive estimate. The important fact is that the bound vanishes as $\lambda \rightarrow 0$.

⁹ The use of k vectors in GS95 stems from the fact that the concept the local system of reference was not introduced till the later works.

the predicted amount (41), $\langle \ell_\perp^2 \rangle \sim \varepsilon t^3$, one finds that

$$\frac{\langle \ell^2(t) \rangle}{\langle \ell^2(t) \rangle_{\text{frozen}}} \sim M_A^2 \left(\frac{t}{t_L} \right)^2 S_L, \quad (42)$$

where $t_L = L_i/u_L$ is the eddy-turnover time and $S_L = v_A L_i/\lambda$ is the Lundquist number based on the turbulence injection scale. For $S_L \gg 1$ typical of astrophysical systems, the violations of standard flux-freezing are enormous and increase with time. It is also not true however that flux-freezing is completely violated. Stochastic flux-freezing in the sense of (31)-(33) remains valid for the “virtual” particle trajectories in a plasma flow which corresponds to a high (kinetic and magnetic) Reynolds-number, turbulent MHD solution.

There is another way to understand spontaneous stochasticity by considering the real trajectories of actual plasma fluid elements, which obey (38). Suppose that one considers two fluid-elements located initially at locations \mathbf{a}, \mathbf{a}' which are displaced from each other by a small amount $\boldsymbol{\rho} = \mathbf{a}' - \mathbf{a}$. At early times the separation of the particles is ballistic (Batchelor 1950). For example, GS95 theory implies that in the transverse direction

$$|\mathbf{x}_\perp(\mathbf{a}', t) - \mathbf{x}_\perp(\mathbf{a}, t)| \sim |\mathbf{u}_\perp(\mathbf{a}') - \mathbf{u}_\perp(\mathbf{a})|t \sim (\varepsilon \rho_\perp)^{1/3} t \quad (43)$$

in the rms sense, for times $t < t_0 = O((\rho_\perp^2/\varepsilon)^{1/3})$ and separations $\rho_\perp > \rho_\nu = O((\nu^3/\varepsilon)^{1/4})$. However, for $t \gg t_0$ the Richardson-type result

$$|\mathbf{x}_\perp(\mathbf{a}', t) - \mathbf{x}_\perp(\mathbf{a}, t)| \sim (\varepsilon t^3)^{1/2}. \quad (44)$$

again holds in the rms sense. Note that the initial particle separation $\boldsymbol{\rho}$ is completely “forgotten” at sufficiently long times! See Fig. 2. In the limit taking first $\nu \rightarrow 0$ and then $\rho \rightarrow 0$, one obtains infinitely many solutions of the deterministic initial-value problem (38). In fact, one can see by taking $\tilde{\mathbf{a}}' = \mathbf{a} + \tilde{\boldsymbol{\rho}}$ for a random displacement $\tilde{\boldsymbol{\rho}}$ that the real fluid particle trajectories also become random in this limit. More precisely, consider an ensemble of random displacements inside the ball $|\tilde{\boldsymbol{\rho}}| < \rho_0$. Then one obtains a random ensemble of real particle trajectories by solving (38) for $\mathbf{x}(\mathbf{a} + \tilde{\boldsymbol{\rho}}, t)$ and taking the limits first $\nu \rightarrow 0$ and then $\rho_0 \rightarrow 0$. In fact, it can be shown using Richardson’s diffusion approximation for 2-particle dispersion that this is the same as the random ensemble of “virtual” particle trajectories obtained by solving (31) and taking the limits $\nu, \lambda \rightarrow 0$ together (E & vanden Eijnden 2000). Put another way, for every “virtual” particle trajectory $\tilde{\mathbf{x}}(\mathbf{a}, t)$ there is a real fluid particle trajectory $\mathbf{x}(\mathbf{a} + \tilde{\boldsymbol{\rho}}, t)$ such that $|\tilde{\boldsymbol{\rho}}| \rightarrow 0$ and $|\tilde{\mathbf{x}}(\mathbf{a}, t) - \mathbf{x}(\mathbf{a} + \tilde{\boldsymbol{\rho}}, t)| \rightarrow 0$ in the limit $\nu, \lambda \rightarrow 0$. “Virtual” particle trajectories coincide with real fluid particle trajectories in this statistical sense.

The breakdown of conventional flux-freezing due to “roughness” of the fields and spontaneous stochasticity of the Lagrangian flows means that magnetic line-topology is no longer preserved in time for the limit $\lambda \rightarrow 0$. The infinite ensemble of “virtual magnetic fields” frozen-in to the stochastic flows all have exactly the same line-topology as the initial magnetic field. However, the average over the ensemble of virtual field-lines that arrive to the same final point which gives the resultant physical magnetic field will, in general, change the topology.

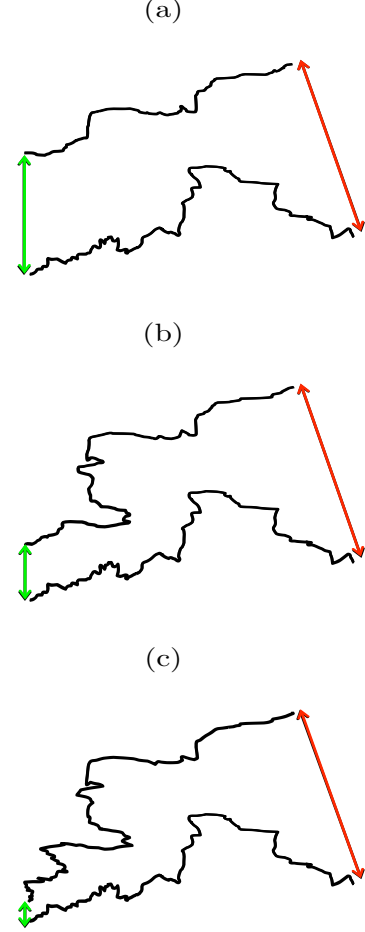


FIG. 2.— *Illustration of spontaneous stochasticity.* Three pairs of Lagrangian trajectories are shown in panels (a),(b),(c), with initial separations at time t_0 between the two members of the pair indicated by green arrows and final separations at time t indicated by red arrows. The initial separations are progressively decreased going from (a) to (b) to (c), but the final separations remain the same! Continuing in this manner, one obtains as a limit the situation with more than one Lagrangian trajectory for the same initial point.

In particular, this permits fast topology change (magnetic reconnection) independent of the precise value of the resistivity. It is worth noting, however, that not all topological conservation laws are vitiated. Magnetic helicity conservation, for example, is particularly robust, even when velocity and magnetic fields are extremely “rough”. It has been proved by Caffisch et al. (1997) [Theorem 4.2] that magnetic helicity is conserved in the limit as $\lambda \rightarrow 0$ even for such “rough” fields, as long as the scaling exponents of velocity and magnetic field 3rd-order structure functions remain positive. This result is consistent with very extreme singularities such as sparsely distributed tangential discontinuities (current and vortex sheets). Thus, fast reconnection due to the mechanism of spontaneous stochasticity will generally conserve magnetic helicity.

One last important remark: everything that we have stated about Lagrangian particle trajectories applies just as well to the magnetic field-lines $\boldsymbol{\xi}(\mathbf{x}, \sigma)$, defined by solv-

ing (at each fixed time t)

$$\begin{cases} \frac{d\xi}{d\sigma}(\mathbf{x}, \sigma) = \mathbf{B}(\xi(\mathbf{x}, \sigma), t), \\ \xi(\mathbf{x}, 0) = \mathbf{x}. \end{cases} \quad (45)$$

Here $\xi(\mathbf{x}, \sigma)$ is the field-line which passes through point \mathbf{x} (at time t). The parameter σ is related to arc-length s along the field-line by $ds = |\mathbf{B}(\xi(\mathbf{x}, \sigma), t)|d\sigma$. In GS95 theory, field lines which correspond to two nearby points \mathbf{x}, \mathbf{x}' with $\mathbf{r} = \mathbf{x}' - \mathbf{x}$ will separate in the transverse direction as

$$|\xi_{\perp}(\mathbf{x}', s) - \xi_{\perp}(\mathbf{x}, s)| \sim (s^3/L_i)^{1/2} M_A^2 \quad (46)$$

in the rms sense, for $s \gg s_0 = O(v_A(r_{\perp}^2/\varepsilon)^{1/3})$ and for $r_{\perp} > r_{\lambda} = O((\lambda^3/\varepsilon)^{1/4})$. See eq.(8). This is exactly the “stochastic line-wandering” which was invoked in the LV99 theory of fast magnetic reconnection and, later, in the theory of thermal conduction in a turbulent MHD plasma (see Narayan & Medvedev (2001) for $M_A = 1$ and Lazarian (2006) for an arbitrary M_A). Just as for Lagrangian trajectories, the initial separation \mathbf{r} between field lines is “forgotten” after proceeding a large enough distance along them. This is the reason that the aforementioned results for the thermal conductivity do not depend upon the electron Larmor radius ρ_e . In fact, magnetic field-lines become stochastic in the limit $\lambda \rightarrow 0$ in precisely the same sense as do Lagrangian particle trajectories in the limit $\nu \rightarrow 0$.¹⁰ Consider a random bundle of field lines $\xi(\mathbf{x} + \tilde{\mathbf{r}}, s)$ for a stochastic displacement vector $\tilde{\mathbf{r}}$ distributed over the ball $|\tilde{\mathbf{r}}| < r_0$ of radius r_0 . By taking the limits first $\lambda \rightarrow 0$ and then $r_0 \rightarrow 0$, one obtains an infinite ensemble of magnetic field-lines all passing through the same point \mathbf{x} !

4.3. The generalized Ohm’s law

Our arguments for the essential stochasticity of turbulent flux-freezing in sections 4.1-2 may have given the impression that the origin of the randomness is necessarily connected with Ohmic resistivity. If our conclusions really did depend upon the resistive MHD model (29) assumed there, then they could be criticized as physically unrealistic. Indeed, there are in actuality many non-ideal terms that appear in the magnetic equations of motion for an ionized plasma, which may be summarized in the generalized Ohm’s law

$$\begin{aligned} \mathbf{E} = & -\frac{1}{c}\mathbf{u} \times \mathbf{B} + \eta_{\perp} \mathbf{J}_{\perp} + \eta_{\parallel} \mathbf{J}_{\parallel} + \frac{\mathbf{J} \times \mathbf{B}}{nec} \\ & - \frac{\nabla \cdot \mathbf{P}_e}{nec} + \frac{m_e}{ne^2} \left(\frac{\partial \mathbf{J}}{\partial t} + \nabla \cdot (\mathbf{u} \mathbf{J} + \mathbf{J} \mathbf{u} - \frac{1}{ne} \mathbf{J} \mathbf{J}) \right) \end{aligned} \quad (47)$$

when quasineutrality and $m_e \ll m_i$ are assumed. See Vasyliunas (1975); Priest & Forbes (2000); Bhattacharjee et al. (1999). The electric fields appearing on the right-hand side are, respectively, the motional field, Ohmic fields associated to perpendicular and parallel resistivities, the Hall field, a contribution from the electron pressure tensor, and electron inertial contributions. All of these terms have been invoked in various theories of fast magnetic reconnection. However, we shall argue that

¹⁰ In the limit of large viscosities that we do not consider here the physics of field wandering is somewhat different and requires a separate discussion (see Lazarian, Vishniac & Cho 2004).

in the turbulent environments that we consider none of these terms alters our fundamental conclusions. In the first place, the stochasticity of flux-freezing in turbulent plasmas is due not to resistivity (or to other terms in the generalized Ohm’s law) but instead to the “roughness” of the MHD solutions in a long inertial range. In the second place, the precise microscopic plasma mechanism of “line-breaking” that acts at small scales (below the ion or electron gyroradii) is irrelevant to the inertial-range turbulence dynamics, which will be fundamentally the same for any such mechanism.

To demonstrate the first point, we show that flux-freezing is effectively stochastic in the turbulent inertial-range even for the ideal induction equation

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{u} \times \mathbf{B}) \quad (48)$$

with *all* non-ideal terms in (47) set to zero! Here we assume that \mathbf{u}, \mathbf{B} are smooth at length-scales below some cut-off $< \ell_d$ but rough (turbulent) at larger scales. Notice that the above equation (48) is realistic at leading-order for collisionless, magnetized plasmas at scales larger than the ion-gyroradius ρ_i (Kulsrud 1983), while the smoothness scale ℓ_d set by field-perpendicular viscosity and resistivity is often of the same order or smaller than ρ_i . Thus, our assumptions are quite realistic. Because of the cut-off ℓ_d , flux-freezing in the standard sense must, in fact, be valid. How then can we claim that it becomes stochastic? To see this, consider the magnetic field observed at some finite space-resolution ρ :

$$\bar{\mathbf{B}}_{\rho}(\mathbf{x}, t) = \int d^3r G_{\rho}(\mathbf{r}) \mathbf{B}(\mathbf{x} + \mathbf{r}, t), \quad (49)$$

where we have introduced a coarse-graining kernel G_{ρ} to represent the smearing effect of the observation over a ball of radius ρ around the space point \mathbf{x} . We shall assume below that $\ell_d \ll \rho \ll L_i$, the scale of the largest turbulent eddies. Thus, $\rho \gtrsim \rho_i$ satisfies these conditions. Applying the standard Lundquist formula for the frozen-in magnetic field, one obtains

$$\bar{\mathbf{B}}_{\rho}(\mathbf{x}, t) = \int \frac{\mathbf{B}(\mathbf{a}, t_0) \cdot \nabla_{\mathbf{a}} \mathbf{x}_{t, t_0}(\mathbf{a})}{\det(\nabla_{\mathbf{a}} \mathbf{x}_{t, t_0}(\mathbf{a}))} \bigg|_{\mathbf{x}_{t, t_0}(\mathbf{a}) = \mathbf{x} + \mathbf{r}} G_{\rho}(\mathbf{r}) \cdot d^3r \quad (50)$$

The Lagrangian particle trajectories that appear in this formula start in the ball of radius ρ around \mathbf{x} at time t and then follow the flow velocity \mathbf{u} backward in time to t_0 , as illustrated in Fig. 3. When ρ lies in a GS95 turbulent inertial range, then the trajectories explosively separate to a perpendicular distance $\Delta x_{\perp} \sim (\varepsilon|t - t_0|^3)^{1/2}$, independent of ρ at times $|t - t_0| \gg (\rho^2/\varepsilon)^{1/3}$. The result is indistinguishable from the stochastic Lundquist formula (35) which was derived in section 4.1 using the stochastic representation of Laplacian resistivity. In fact, in a formal mathematical limit taking first $\ell_d \rightarrow 0$, then $\rho \rightarrow 0$, the Lagrangian trajectories in (50) remain stochastic and the two formulas coincide. This is a rigorous theorem for the Kazantsev-Kraichnan dynamo model, where it has been proved that the ensemble of stochastic Lagrangian trajectories as constructed above is precisely the same as that obtained for the $\lambda \rightarrow 0$ limit (E & vanden Eijnden 2000). Stochasticity of flux-freezing is not due intrinsically to resistivity or other microscopic plasma mech-

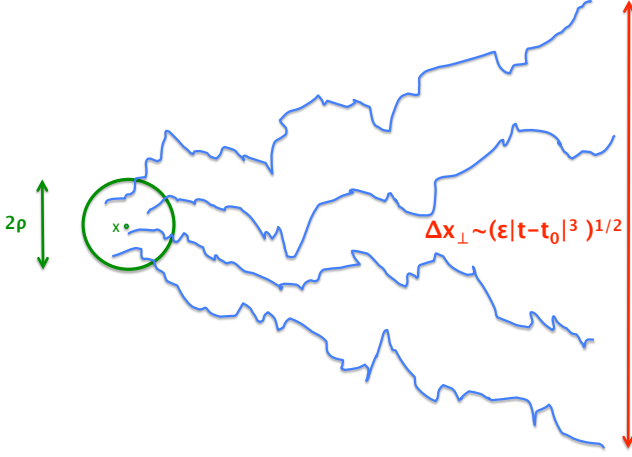


FIG. 3.— Lagrangian trajectories that start in the ball of radius ρ around space point \mathbf{x} at time t move backward in time to t_0 , explosively separating to a field-perpendicular distance $\Delta x_{\perp} \sim (\varepsilon|t-t_0|^3)^{1/2}$ which is independent of ρ for $|t-t_0| \gg (\rho^2/\varepsilon)^{1/3}$.

anisms that “break” field-lines but is, instead, a fundamental consequence of turbulent Richardson diffusion.

Higher-order terms in the generalized Ohm’s law (47) that do not appear in the ideal equation (48) will lead to melding and merging of field-lines at scales $< \rho_i$. However, the above argument strongly suggests that these details of the microscopic plasma processes do not affect the dynamics at scales larger than ρ_i . In some cases this can be shown more analytically by defining a suitable “motion” of field-lines consistent with the induction equation. For example, the formulation in section 4.1 based on addition of a Brownian motion to the Lagrangian particle dynamics, eq.(31), can be carried over to certain instances of the generalized Ohm’s law. See in particular Eyink (2009) for the Hall MHD equations. This approach is used in Appendix B to argue that neither the Hall effect nor Ohmic resistivity will have any significant influence on the inertial-range turbulence dynamics at large enough scales. The Hall term, for example, does not affect the dynamics at scales much greater than $\delta_i = \rho_i/\sqrt{\beta_i}$, the ion skin depth. Unfortunately, it is difficult to extend this type of argument to all cases of the generalized Ohm’s law, because it is not known how to define a “motion” of field-lines consistent with the induction equation for the general case.

On the other hand, there is a different argument which applies in general and leads to the same conclusion that flux-freezing must be intrinsically stochastic in turbulent plasmas. While the “motion” of magnetic field-lines is a conventional and somewhat arbitrary concept, the motion of plasma is perfectly well-defined within the validity of an MHD description. Plasma fluid moves with the bulk velocity \mathbf{u} . Thus, field-lines may be tracked by “tagging” the lines with plasma fluid elements and then following these as Lagrangian fluid particles (Newcomb 1958; Axford 1984). In the case of a smooth, laminar solution of the ideal MHD equations, this is unambiguous because of Alfvén’s theorem: two plasma particles which start on a certain field-line must share a field-line for all times. One can then, by convention, consider this as the “same” field line as the initial one. This approach fails

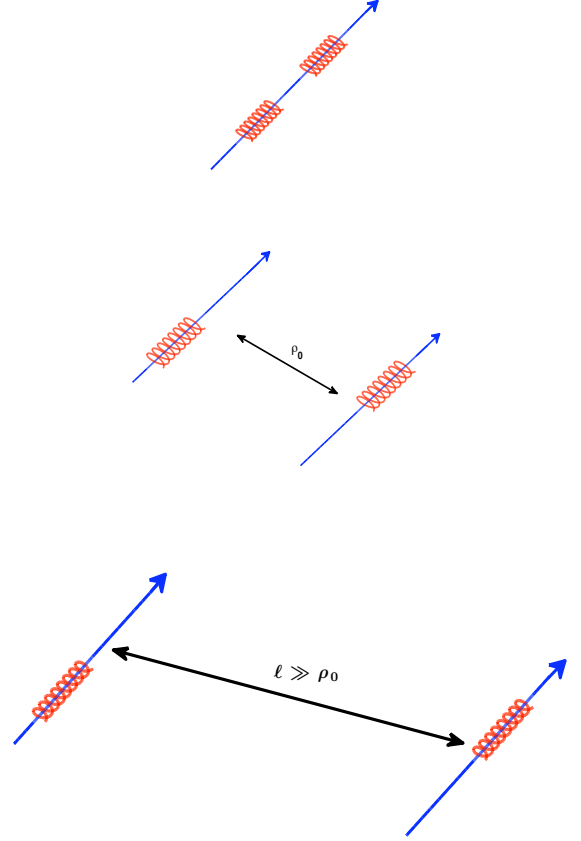


FIG. 4.— *Advection acceleration of magnetic field-line separation.* *Top:* A pair of ions starting on the same field-line. *Middle:* Electric field in the frame of the plasma changes the connections of the ions to field-lines separated by distance ρ_0 . *Bottom:* The ions are swept apart by fluid motions to a distance $\ell \gg \rho_0$ in a time t_ℓ that is independent of ρ_0 . The original field-line has effectively “split” into two separated by the macroscopic distance ℓ .

for a non-ideal Ohm’s law,

$$\mathbf{E} + \frac{1}{c} \mathbf{u} \times \mathbf{B} = \mathbf{R}, \quad (51)$$

where \mathbf{R} represents all of the terms on the righthand side of (47) other than the motional term. Clearly, \mathbf{R} is just the electric field in the rest frame of the plasma flowing with the bulk velocity \mathbf{u} . Emf due to these non-ideal terms leads to time-dependent magnetic flux in the rest frame, corresponding to a slippage of field lines. This vitiates the usual method to assign an identity to individual field-lines over time, because plasma elements shift their attachments to lines. Charged particles move along magnetic field-lines, but two plasma elements that start on one field-line will sit on distinct field-lines at later times.

Now consider a turbulent plasma where the non-ideal term is numerically “small” but the plasma has a turbulent inertial-range in which the velocity field \mathbf{u} is rough, with a power-law energy spectrum extending down to a smallest length-scale $\rho_0 \approx \rho_i$. The slight shifts in line-attachments are enormously amplified by explosive rela-

tive advection, as illustrated in Fig. 4. Consider a single magnetic field-line in this plasma and, along it, two plasma fluid particles at initial locations \mathbf{a} and \mathbf{a}' . Due to a combination of the non-ideal field \mathbf{R} and advection by sub-inertial-range eddies, the two plasma particles will end up on *distinct* field-lines displaced a distance $|\mathbf{x}(\mathbf{a}', t) - \mathbf{x}(\mathbf{a}, t)| = \rho_0$ apart in a time τ_0 which is generally microscopically small compared with the eddy-turnover time t_L . Because of (44), the two plasma elements will subsequently end up on field-lines displaced a macroscopic distance $(\varepsilon t^3)^{1/2}$ apart independent of ρ_0 in a time $t \gg \tau_0$ ¹¹. In effect, the original field-line has split apart into two lines separated by macroscopic distance in a finite time t . Of course, this same argument works if we take $\mathbf{a}' = \mathbf{a} + \tilde{\boldsymbol{\rho}}$ for some small random displacement $\tilde{\boldsymbol{\rho}}$ along the initial field-line with $|\tilde{\boldsymbol{\rho}}| < \rho_0$. We have already argued in section 4.2 that the random ensemble of Lagrangian trajectories $\mathbf{x}(\mathbf{a} + \tilde{\boldsymbol{\rho}}, t)$ will coincide for $t \gg \tau_0$ with the random ensemble $\tilde{\mathbf{x}}(\mathbf{a}, t)$ obtained from the perturbation by Brownian motion in eq.(19). The initial field-line will be stochastically “frozen-in” to this ensemble of random flows.

Turbulent advection accelerates the separation of field-lines so effectively that the microscopic plasma process of line-slippage, whatever its origin, is rapidly “forgotten”. The non-ideal terms in the generalized Ohm’s law become insignificant in further separating lines which have exceeded the distance ρ_0 apart. When the plasma flow velocity \mathbf{u} is “rough”, it is the motional term in the generalized Ohm’s law which, after a very short time, dominates all of the other terms in the separation of field-lines.

5. LARGE-SCALE MAGNETIC RECONNECTION

With the theory developed above, we now turn to our main topic of magnetic reconnection. To be specific, we consider the steady-state reconnection of a pair of large-scale magnetic flux tubes which collide along a section of length L_x . We assume incompressible plasma flow and anti-parallel magnetic flux-tubes of equal field-strength $|B_x|$. The situation could be generalized to allow for compressible plasma and asymmetric field-strengths (Cassak & Shay 2007), flux-tubes intersecting at an angle or with a shared component of magnetic field (guide field) B_z (Linton et al. 2001), twisted flux-tubes with magnetic helicity (Zweibel & Rhoads 1995), etc. All of these are relevant for applications, but we keep the set-up simple in order to make the new ideas clear. The large-scale geometry we consider is thus precisely the two-dimensional configuration in classical Sweet-Parker theory. However, we are working in three-space dimensions and thus small-scale turbulence, in particular, will be fully 3D.

5.1. A Rederivation of LV99 by Stochastic Flux-Freezing

The constraints on steady-state reconnection from the MHD balance equations are well-known. For a reconnection layer of length L_x and thickness Δ , the constraint

¹¹ In the conventional view, for ideal plasmas with $\mathbf{R} \equiv \mathbf{0}$, the two elements would separate also to this distance but would remain on the same, highly-stretched field-line. This is not the case with a nonideality $\mathbf{R} \neq \mathbf{0}$, however small.

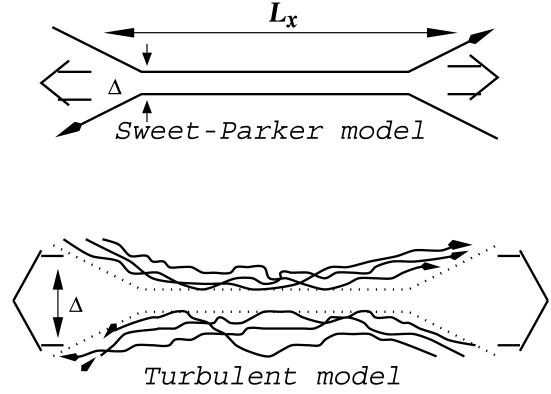


FIG. 5.— *Upper plot:* Sweet-Parker model of reconnection. The outflow is limited by a thin slot Δ , which is determined by Ohmic diffusivity. The other scale is an astrophysical scale $L_x \gg \Delta$. *Lower plot:* Reconnection of weakly stochastic magnetic field according to LV99. The model that accounts for the stochasticity of magnetic field lines. The outflow is limited by the wandering of magnetic field lines, which depends on field line stochasticity. From Lazarian et al. 2004.

of energy conservation is

$$\frac{1}{2}\rho v_{out}^2 v_{out} \Delta = \frac{1}{8\pi} B_x^2 v_{rec} L_x, \quad (52)$$

where v_{rec} is the in-flow plasma speed or reconnection velocity and v_{out} is the out-flow velocity from the sides of the reconnection layer. Here it is assumed that the energy in the out-flow jets is primarily kinetic, since the B_y component of the outgoing field-lines at the outer edges of the reconnection layer is still relatively small. The critical constraint of mass conservation is

$$v_{out} \Delta = v_{rec} L_x. \quad (53)$$

Combining (52) and (53) yields

$$v_{out} = \frac{B_x}{\sqrt{4\pi\rho}} = v_A, \quad (54)$$

so that the out-flow velocity is at the Alfvén speed of the in-coming magnetic fields. The momentum equation enforces pressure balance along and across the reconnection layer and does not yield a new constraint in our setting. From (53) and (54) we see that

$$v_{rec} = v_A \frac{\Delta}{L_x}. \quad (55)$$

Our discussion to this point has been completely general and no particular mechanism of line-slippage has been assumed. To proceed further, a specific reconnection mechanism must be invoked which permits one to estimate the thickness Δ of the layer.

The classical Sweet-Parker model assumes resistive reconnection in a laminar MHD solution. There are then several ways to estimate the thickness Δ , which are all consistent. The most common approach is to use the steady-state Faraday’s law, which implies the constancy in the vertical y -direction of the reconnection electric field E_z . Outside the reconnection layer $E_z = v_{rec} B_x / c$,

while inside $E_z = \eta J_z = \eta c B_x / 4\pi \Delta$. Equating these gives

$$v_{rec} = \frac{\lambda}{\Delta}, \quad (56)$$

where $\lambda = \eta c^2 / 4\pi$ is the magnetic diffusivity. Combining (55) and (56) gives the well-known Sweet-Parker results that

$$\Delta = L_x / \sqrt{S}, \quad v_{rec} = v_A / \sqrt{S}, \quad (57)$$

where $S = v_A L_x / \lambda$ is the Lundquist number. Another way to obtain this result is based on standard flux-freezing ideas for a laminar plasma flow (Kulsrud, 2005). The mean-square vertical distance that a magnetic field-line can diffuse by resistivity in time t is

$$\langle y^2(t) \rangle \sim \lambda t. \quad (58)$$

Each reconnected field-line is the result of resistive gluing of field-lines that diffuse vertically across the reconnection layer. At the same time, the field lines are advected out of the sides of the reconnection layer of length L_x at a velocity of order v_A . Thus, the time that the lines can spend in the resistive layer is the Alfvén crossing time $t_A = L_x / v_A$. Thus, field lines can only be merged that are separated by a distance

$$\Delta = \sqrt{\langle y^2(t_A) \rangle} \sim \sqrt{\lambda t_A} = L_x / \sqrt{S}. \quad (59)$$

We thus recover (57). This is an important dynamical consistency check on the Sweet-Parker model of laminar resistive reconnection.

For large Lundquist numbers, the Sweet-Parker reconnection speed (57) is too small to account for observed reconnection rates in most astrophysical settings. However, the Sweet-Parker laminar MHD solution is not appropriate for turbulent plasmas at large S . For simplicity, we shall consider reconnection in the presence of a steady-state background turbulence with an integral length-scale L_i and rms velocity u_L . We shall consider sub- and trans-Alfvénic turbulence with $u_L \leq v_A$ or $M_A \leq 1$, under the assumption that the large-scale magnetic flux tubes are strong and well-defined in the presence of turbulence. In some situations the reconnection may be initially laminar and then turbulence initiated by instabilities triggered by small fluctuations from the environment. Initial fast reconnection by Hall effect (Shay et al. 1998; Wang et al. 2000; Birn et al. 2001; Drake 2001) or other collisionless processes may itself provide the energy release to generate small-scale MHD turbulence. We shall consider only the final stage of steady-state reconnection in the presence of turbulence with the given characteristic length and velocity scales.

The key issue is to estimate Δ . We shall give two different methods of determining the width of the turbulent reconnection layer.

Our first estimate is based on stochastic flux-freezing/SFF (section 4) and 2-particle turbulent diffusivity (section 3.2). We have seen that GS95 theory implies that the turbulent separation of pairs of “virtual” field-lines backward in time, in the y -direction perpendicular to the mean magnetic field, is given by eqs.(23) and (28), or

$$\langle y^2(t) \rangle \sim \begin{cases} \varepsilon t^3, & t_\eta \ll t < t_c \\ u_L L_i M_A^3 t, & t > t_c, \end{cases} \quad (60)$$

for the strong and weak MHD turbulence regimes, respectively, with $t_c = L_i / v_A$. Reconnected field-lines that emerge from the turbulent reconnection layer are constituted from field-lines obtained by following “virtual” field lines backward in time over the Alfvén crossing period $t_A = L_x / v_A$. We thus obtain $\Delta = \sqrt{\langle y^2(t_A) \rangle}$ as the width of the turbulent reconnection layer. There are two cases to consider, depending upon whether $t_A < t_c$ or $t_A > t_c$. Using $\varepsilon \sim u_L^4 / v_A L_i$, we rewrite (60) as

$$\langle y^2(t) \rangle \sim \begin{cases} (L_x^3 / L_i) M_A^4 (t/t_A)^3 & \text{if } t < t_c \\ L_i L_x M_A^4 (t/t_A), & \text{if } t > t_c. \end{cases} \quad (61)$$

Since $t_c \leq t_A$ exactly when $L_i \leq L_x$, respectively, we get finally that

$$\Delta = L_x M_A^2 \min \left\{ \left(\frac{L_x}{L_i} \right)^{1/2}, \left(\frac{L_i}{L_x} \right)^{1/2} \right\}. \quad (62)$$

This is precisely the result of Lazarian & Vishniac (1999). The present derivation avoids many complications in their original argument due to the fact that “virtual” field lines pass directly through each other. LV99 had to consider the limit to the reconnection rate set by the speed with which reconnected magnetic field elements can pass through each other on the way out of the reconnection zone, employing a plausible but heuristic self-consistency argument. This problem is avoided completely and rigorously by using SFF (which is mathematically equivalent to the resistive MHD equations).

The thickness Δ of the turbulent reconnection layer was originally estimated in LV99 by a geometric argument, based upon the spontaneous stochasticity of the magnetic field-lines themselves. The plasma in the reconnection zone can only be expelled along the lines of the (real, physical) magnetic field. As we have seen in section 3.1, the lateral wandering as one moves a distance s along a field line is given by eqs.(8), (9), or

$$\langle y^2(s) \rangle \sim \begin{cases} (s^3 / L_i) M_A^4 & \text{if } s < L_i \\ L_i s M_A^4, & \text{if } s > L_i. \end{cases} \quad (63)$$

Since the plasma must move a distance $s \sim L_x$ along the field-line in order to escape the reconnection layer, we can estimate the thickness of the layer by $\Delta \sim \sqrt{\langle y^2(L_x) \rangle}$. Using (63) we immediately recover (62). The agreement of these two arguments demonstrates some basic dynamical consistency of the LV99 theory.

We obtain from both arguments the LV99 estimate for the reconnection speed

$$v_{rec} = v_A M_A^2 \min \left\{ \left(\frac{L_x}{L_i} \right)^{1/2}, \left(\frac{L_i}{L_x} \right)^{1/2} \right\}. \quad (64)$$

Note that $v_{trans} = v_A M_A^2$ is the amplitude of velocity fluctuations at the transition between weak and strong MHD regimes. This can be a sizable fraction of the Alfvén velocity. Furthermore, (64) is “fast” in the strictest sense of being independent of the microscopic resistivity. The physical mechanism of this independence is spontaneous stochasticity, both of the Lagrangian particle trajectories and of the magnetic field lines. In both cases, initial separations by a microscopic resistive length are amplified to macroscopic distances as one traverses

along the wandering curves by an amount (of time or of path-length, respectively) which is independent of the microscopic electrical resistivity.

The above discussion has approximated the reconnection layer as a zone of homogeneous MHD turbulence, with a constant Alfvénic Mach number $M_A = u_L/v_A$ set by the reconnecting x -component of the upstream magnetic field, $v_A = B_x/\sqrt{4\pi\rho}$. In reality, there will be a “magnetic shear layer” with a continuous change of the mean magnetic field $\bar{B}_x(y)$ across the layer. Likewise, there are nontrivial profiles $\bar{u}_y(x, y)$ and $\bar{u}_x(x, y)$ of both incoming and outgoing velocities, which are known to have nontrivial effects on 2-particle dispersion (Shen & Yeung 1997; Pope 2002). An interesting refinement of the original LV99 theory would be to consider the effect of variation of the mean velocity and magnetic fields across the reconnection layer. One plausible effect of magnetic shear will be to increase M_A within the layer, permitting greater lateral wandering of field-lines and increasing the thickness Δ . Less easy to guess are the effects of the change in orientations of magnetic fields for flux-tubes reconnecting at an angle of less than 180° (Linton et al. 2001). The simulation results of Kowal et al. (2009) on tubes crossed at various angles give support to the idea that the effects are minimal, but further theoretical and numerical study of this would be desirable. Another effect which has been neglected in our derivation above is the turbulent energy dissipation that exists in the reconnection layer. This can be estimated as $\rho\varepsilon\Delta L_x$ with $\varepsilon = u_L^4/v_A L_i$. If this is added into the energy balance (52) one obtains a result for the outflow velocity

$$v_{out}^2 = v_A^2 \left(1 - 2M_A^4 \frac{L_x}{L_i} \right) \quad (65)$$

which can be substantially reduced from the Alfvén speed. By the mass conservation constraint (53), there is a similar reduction in the reconnection speed. However, these effects are ignorable under the condition $M_A^4 \ll L_i/L_x$ or, equivalently, $\Delta \ll L_i$. This is generally well-satisfied. Other modifications and refinements of the LV99 model can be envisaged and are worth pursuing (see the opening paragraph of this section and also section 6.4).

It is important to emphasize that the original LV99 argument made no essential use of averaging over turbulent ensembles. The stochastic line-wandering which is the essence of their argument holds in every realization of the flow, at each instant of time. The “spontaneous stochasticity” of field lines is not a statistical result in the usual sense of turbulence theory and does not arise from ensemble-averaging. The only use of ensembles in LV99 is to get a measure of the “typical” wandering distance Δ of the field lines (in an rms sense). If one looks at different ensemble members of the turbulent flow, or at different single-time snapshots of the steady reconnection state, then Δ will fluctuate. Thus, the reconnection rate will also fluctuate a considerable amount over ensemble members or over time. E.g. see Figs. 12-14 in Kowal et al. (2009). But it will be “fast” in each realization and at each instant of time, because the mass outflow constraint is lifted by the large wandering of field lines.

Exactly the same remarks apply to the rederivation of LV99 using stochastic flux-freezing (SFF) and Richard-

son diffusion. SFF is also not “stochastic” in the usual sense of turbulence theory. The new derivation uses ensemble-averaging just to get an estimate of the “typical” wandering of stochastic Lagrangian trajectories in an rms sense. Numerical studies of Richardson’s t^3 law in hydrodynamic turbulence show that it is a very robust property. At most points in a turbulent flow, a “cloud” of stochastic particles spreads with a power-law very close to t^3 . There is some intermittency effect, with certain rarer, high-intensity points showing a much faster dispersion or low-intensity a slower dispersion, but it is not a huge effect. Only the higher-order moments of the relative separation distance are strongly affected (Boffetta & Sokolov 2002).

The LV99 theory, therefore, does not involve “turbulent resistivity” or “turbulent magnetic diffusivity” as this is usually understood. This is ordinarily meant to be an enhanced diffusivity experienced by the ensemble-averaged magnetic field $\langle \mathbf{B} \rangle$. However, it is only if one assumes some scale-separation between the mean and the fluctuations that the effect of the fluctuations can be legitimately described as an enhanced diffusivity (Moffatt 1983). In realistic turbulent flows, with no scale-separation, this phenomenological description as an effective diffusivity can be wildly inaccurate. LV99 makes no appeal to such concepts and, indeed, never considers the ensemble-average field $\langle \mathbf{B} \rangle$ at all. Neither does Nature. The Sun does not average over an ensemble of coronal loops in order to get fast reconnection!

5.2. Small-Scale Reconnection and MHD Turbulence

The focus of this paper has been on the reconnection of large-scale, oppositely-directed magnetic flux-tubes in a turbulent plasma, since it is this phenomenon which has the most direct interest for astrophysics. On the other hand, a similar process of reconnection of local magnetic field-lines must take place at small-scales in homogeneous MHD turbulence, with or without a background magnetic field. It has been argued that such small-scale magnetic reconnection is an essential feature of MHD turbulence (Matthaeus & Lamkin 1986; Lazarian & Vishniac 1999). A substantial literature has since arisen on the relation between MHD turbulence and magnetic reconnection (e.g. see Lazarian (2005); Eyink & Aluie (2006); Servidio et al. (2010); Mininni & Pouquet (2009)). We cannot review this entire subject here, but it is important to discuss briefly the implications of “spontaneous stochasticity” for the problem.

It was observed already in LV99 that nearly parallel field-lines in adjacent turbulent eddies must frequently intersect and reconnect according to GS95 theory. The local geometry is similar to that in large-scale reconnection “writ small”, with the local mean magnetic field acting as a guide-field and the field-lines with anti-parallel transverse components undergoing reconnection. The crossing-points and knots in the local magnetic field-lines, if unresolved, would create topological obstructions to the plasma flow which would render hydrodynamic motion impossible. LV99 argued instead that line-intersections at perpendicular length-scale ℓ_\perp are resolved by magnetic reconnection in the eddy-turnover time $\ell_\perp/\delta u(\ell)$, independent of the resistivity. Fast magnetic reconnection is thus crucial for the very existence of MHD turbulence of the kind supposed in GS95.

The stochastic “line-wandering” implied by GS95 theory (section 3.1) suggests that a very complex line-topology must exist in turbulent flows with magnetic fields that are non-smooth on inertial-range scales. This complexity is made particularly evident by the “spontaneous stochasticity” phenomenon (section 4.2), according to which there are infinitely-many field-lines passing through each point in the limit of zero resistivity! This result suggests that reconnection sites are distributed densely in space for MHD turbulence at very high conductivity. Turbulent plasma flows with rough magnetic and velocity fields have a very strange and intricate geometry when compared with the smooth, laminar flows that have often been the focus of theoretical and empirical studies of reconnection. Another indication of this complexity is the fractal distribution of clustered magnetic nulls for a magnetic field with a turbulent, power-law energy spectrum (Albright 1999). This latter fact is very intimately connected with the stochastic “line-wandering” phenomenon deduced in LV99, since the magnetic nulls form natural sites of separation of adjacent magnetic field-lines. See Davila & Vassilicos (2003); Goto & Vassilicos (2004) for a discussion of the closely related hydrodynamic phenomenon associated with Lagrangian particle trajectories and stagnation points of the turbulent velocity field.

The above facts strongly suggest that magnetic reconnection must be a process that is active always and everywhere in homogeneous MHD turbulence. As one measure of the local reconnection rate one may take the estimates (23),(28) of the lateral diffusion of magnetic field-lines (for strong and weak turbulence, resp.) For example, in the strong turbulence regime, field lines diffuse in an eddy-turnover-time $\tau_\ell = \varepsilon^{-1/3} \ell_\perp^{2/3}$ at scale ℓ_\perp by a mean-square perpendicular distance $\varepsilon \tau_\ell^3 = \ell_\perp^2$. Thus, reconnection rates are exactly large enough to resolve knots of field-lines at length-scale ℓ_\perp in the natural evolution time at that scale, as argued in LV99. Note that $d\ell_\perp/dt \simeq \delta u(\ell)$ and thus the reconnection speed of pairs of field-lines separated by inertial-scale separations ℓ_\perp is equal to the relative velocity $\delta u(\ell)$ between the two lines.

Another way to quantify small-scale reconnection in MHD turbulence is by violations of Alfvén’s Theorem at length-scale ℓ . This was the approach adopted by Eyink & Aluie (2006), who studied the MHD induction equation in a coarse-grained description at the (isotropic) length-scale ℓ^{12} :

$$\partial_t \bar{\mathbf{B}}_\ell = \nabla \times [\bar{\mathbf{u}}_\ell \times \bar{\mathbf{B}}_\ell + c \boldsymbol{\varepsilon}_\ell - \lambda \nabla \times \bar{\mathbf{B}}_\ell]. \quad (66)$$

Simple estimates show that the direct resistive contribution proportional to λ is negligible at high magnetic Reynolds numbers whenever the magnetic energy stays finite in that limit. See Eyink (2005). It is the turbulent subscale EMF $\boldsymbol{\varepsilon}_\ell = (\mathbf{u} \times \mathbf{B})_\ell - \bar{\mathbf{u}}_\ell \times \bar{\mathbf{B}}_\ell$ which provides the necessary electric field for reconnection at the scale ℓ . The magnetic flux through a material loop $\bar{\mathcal{C}}_\ell(t)$ advected by

the coarse-grained velocity $\bar{\mathbf{u}}_\ell$ satisfies

$$\frac{d}{dt} \oint_{\bar{\mathcal{C}}_\ell(t)} \bar{\mathbf{A}}_\ell \cdot d\mathbf{x} = c \oint_{\bar{\mathcal{C}}_\ell(t)} \bar{\boldsymbol{\varepsilon}}_\ell \cdot d\mathbf{x}. \quad (67)$$

Eyink & Aluie (2006) established necessary conditions for a non-vanishing line-voltage on the righthand side of the above equation, independent of ℓ . These conditions are: either (i) a non-rectifiable (fractal) loop $\bar{\mathcal{C}}_\ell(t)$, or (ii) a blow-up of \mathbf{u} or \mathbf{B} along the loop, or (iii) tangential discontinuities of both \mathbf{u} and \mathbf{B} intersecting the loop $\bar{\mathcal{C}}_\ell(t)$ over a finite length. One or more of these conditions is plausibly satisfied in turbulent flow, implying ubiquitous reconnection. An important remark is that MHD turbulence is dominated by scale-local interactions under very general assumptions. See Eyink (2005); Aluie & Eyink (2010). In particular, the turbulent EMF $\boldsymbol{\varepsilon}_\ell$ is dominated by contributions from velocity and magnetic field modes around the scale ℓ . It is the EMF from these modes which is responsible for reconnection of the lines of the coarse-grained magnetic fields at length-scale ℓ and microscopic plasma processes at smaller scales play no direct role.

5.3. 2D MHD Turbulence and Reconnection

While astrophysical reality is three-dimensional (3D), there is a scholarly and intellectual value to considering MHD turbulence in two-dimensional (2D) space. Comparison of toy models with reality is often a good way to test and sharpen one’s understanding. Numerical simulations can also be better resolved in 2D than in 3D, permitting study of higher kinetic and magnetic Reynolds-number flows. Furthermore, the conventional view is that MHD turbulence in 2D and in 3D are very similar, much more so than for hydrodynamics of neutral fluids. 3D MHD possesses the three ideal, quadratic invariants of total energy, cross-helicity and magnetic helicity and 2D MHD has a nearly identical set of ideal invariants, with magnetic helicity replaced by the mean-square magnetic potential. Absolute equilibrium spectra in 3D suggest that energy and cross-helicity cascade to high-wavenumbers and magnetic helicity to low-wavenumbers (Frisch et al. 1975), while the same argument in 2D leads to an identical conclusion, with an inverse cascade of magnetic potential replacing that of magnetic helicity (Fyfe & Montgomery 1976). Numerical simulations have corroborated these predictions, with the most recent 2D MHD simulations giving energy spectra close to $k^{-5/3}$ or $k^{-3/2}$ in the forward cascade range (Gomez et al. 1999; Biskamp & Schwarz 2001; Ng et al. 2003). On the other hand, from the point of view of GS95 theory, 2D MHD turbulence appears as a very degenerate case. In particular, shear-Alfvén waves that play the dominant role in 3D MHD turbulence according to GS95 are entirely lacking in 2D, where only pseudo-Alfvén wave modes exist. This fact suggests that the character of MHD turbulence in 2D must be quite different than in 3D and certainly not well-described by GS95.

Because of the numerical advantages, many turbulent reconnection studies have also been performed in 2D, beginning with the important work of Matthaeus & Lamkin (1985, 1986). The numerical evidence is rather controversial, however. Matthaeus & Lamkin (1985) reported the reconnection of magnetic islands at rates nearly independent of resistivity in the early stages of their simu-

¹² The field-perpendicular variations therefore include contributions from eddies with ℓ_\perp down to ℓ but the field-parallel variations get contributions from somewhat smaller eddies with ℓ_\perp down to $M_A^2 \ell^{3/2} / L_i^{1/2}$.

lations. Servidio et al. (2010) have more recently made a similar study of Ohmic electric fields at X points in homogeneous, decaying 2D MHD turbulence. This is really a case of small-scale magnetic reconnection, as considered in our previous section 5.3, and not directly relevant to the issue of reconnection of large-scale flux tubes. Two other numerical studies have recently been made of large-scale magnetic reconnection in 2D, by Loureiro et al. (2009) and Kulpa-Dybel et al. (2010), which reach different conclusions. On the one hand, Loureiro et al. (2009) had a better resolution but used periodic boundary conditions, which strongly constrain the ability to do averaging of the reconnection rate and the attainment of the steady state for reconnection. They inferred from their data that the 2D turbulent reconnection rate may be independent of resistivity. On the other hand, Kulpa-Dybel et al. (2010) used smaller data cubes but longer averaging, which is enabled by their outflow boundary conditions. They concluded that the reconnection does depend on resistivity and therefore is slow. The raw data of the two groups are actually rather similar and higher Lundquist-number simulations are probably necessary to resolve the matter.

What does theory say? In particular, can LV99 or some LV99-like theory apply in 2D¹³? At first thought, one might doubt that LV99 is relevant at all since its central element—stochastic line-wandering—seems ruled out in 2D on the simple grounds that magnetic field-lines may not cross. However, closer consideration shows that this argument is fallacious. Magnetic field-lines *can* cross in 2D, at very special points, the magnetic nulls. Furthermore, in a rough magnetic field, the magnetic nulls and null-null lines will form a very complex, fractal web (Albright 1999) which may permit quite unconstrained wandering of field-lines. Indeed, the phenomenon of spontaneous stochasticity of field-lines ought to occur in 2D as well as in 3D, if the magnetic field is rough. There are rigorous examples of this type in 2D, where integral curves of a non-smooth vector field are non-unique at every point in space (e.g. see section II.5 and Fig.2 in Hartman (2002).) From the Lagrangian point of view of Richardson diffusion and stochastic flux-freezing, there is also little difference between 2D and 3D. The Richardson t^3 -law has been observed, for example, in the inverse-energy cascade range of 2D hydrodynamic turbulence (Celani & Vergassola 2001; Goto & Vassilicos 2004) and its analogue could be expected in 2D MHD turbulence. Indeed, if velocity fields are sufficiently rough in 2D MHD turbulence, then spontaneous stochasticity of Lagrangian trajectories ought to hold in 2D just as in 3D, and an LV99-like theory ought to apply.

The above arguments are, however, quite delicate. As a counterexample, we may consider weak MHD turbulence in 2D. The most important difference from 3D is that only pseudo-Alfvén wave modes exist in 2D. Because the weak turbulence cascade preserves $k_{\parallel} = 1/L_i$ while increasing k_{\perp} , the total wavevector is nearly perpendicular and the pseudo-Alfvén polarization vector nearly parallel

to the magnetic field. Our estimate from section 3.2 for the 2-particle diffusivity of weak MHD turbulence is thus reduced in the field-perpendicular direction by a factor of $(\ell_{\perp}/\ell_{\parallel})^2$. The result is that

$$D_{\perp}(\ell) \sim \left(\frac{\ell_{\perp}}{\ell_{\parallel}}\right)^2 \frac{\varepsilon}{\omega_A^2} = \ell_{\perp}^2 \frac{\varepsilon}{v_A^2}. \quad (68)$$

Solving for the perpendicular particle separation from $d\ell_{\perp}^2/dt = D_{\perp}(\ell)$ gives an exponential growth proportional to the initial separation

$$\ell_{\perp}^2(t) \simeq \ell_{\perp}^2(0) \exp((\text{const.})\varepsilon t/v_A^2), \quad (69)$$

i.e., no spontaneous stochasticity! There will be no 2D fast reconnection if such a weak turbulence cascade persists down to dissipation scales. Naively, one would expect this to be true in 2D from a simple time-scale argument. The interaction time of a pair of pseudo-Alfvén waves with parallel length-scale ℓ_{\parallel} and fluctuation amplitude δu_{ℓ} is $\ell_{\parallel}/\delta u_{\ell}$, which will in general be longer than the Alfvén wave-period ℓ_{\parallel}/v_A . This argument suggests that there is no strong MHD turbulence at all in 2D and thus no fast reconnection.

The only way to avoid this conclusion that we can see is that the 2D turbulence develops very singular structures or “tangential discontinuities” with $\delta u_{\ell} \simeq v_A$, independent of ℓ over the inertial range. Note that the rigorous result of Eyink & Aluie (2006) for the 2D case requires such discontinuities and/or blowup in the magnitudes of the velocity or magnetic fields. There seems also to be support for this scenario from numerical simulations of 2D MHD turbulence, which show very intermittent current and vortex sheets in a background of weaker turbulence. It is possible that this is one of those instances where the weak-turbulence cascade is unstable to the formation of localized, singular structures (cf. Majda et al. (1997); Rumpf et al. (2009)). The crucial issue seems to be whether these intermittent structures occur sufficiently densely in spacetime. We shall not attempt to resolve this issue here, but only conclude that the existence of strong MHD turbulence and fast turbulent reconnection in 2D are quite open questions.

5.4. *Alternative Ideas on the Role of Turbulence in Magnetic Reconnection*

As we mentioned in the introduction, various ideas how turbulence can increase the reconnection rate were discussed as far back as 40 years ago. However, these ideas fell short of solving the problem. For instance, some papers have concentrated on the effects that turbulence induces on the microphysical level. In particular, Speiser (1970) showed that in collisionless plasmas the electron collision time should be replaced with the electron retention time in the current sheet. Also Jacobson & Moses (1984) proposed that the current diffusivity should be modified to include the diffusion of electrons across the mean field due to small scale stochasticity. All these effects can only marginally change the reconnection rates.

The closest predecessor to LV99 was the important work of Matthaeus & Lamkin (1985, 1986) (ML85,86). Those authors studied 2D magnetic reconnection in the presence of external turbulence, both theoretically and numerically. They emphasized the very close analogies between the magnetic reconnection layer at high

¹³ In a completely different class are recent approaches which attribute fast reconnection in 2D MHD to formation and merging of plasmoids *without* the presence of conventional turbulence in the current sheet (Samtaney et al. 2009; Cassak et al. 2009; Bhattacharjee et al. 2009; Huang & Bhattacharjee 2010). These will be discussed later in section 6.3.

Lundquist numbers and homogeneous MHD turbulence. They also pointed out various turbulence mechanisms that would enhance reconnection rates, including multiple X-points as reconnection sites and motional EMF of magnetic bubbles advecting out of the reconnection zone. However, ML85,86 did not understand the importance of “spontaneous stochasticity” of field lines and of Lagrangian trajectories and they did not arrive at an analytical prediction for the reconnection speed. An enhancement of the reconnection rate was reported in their numerical study, but the setup precluded the calculation of a long term average reconnection rate. A more recent study along the approach in Matthaeus & Lamkin (1985) is one in Watson et al. (2007), where the effects of small scale turbulence on 2D reconnection were studied and no significant effects of turbulence on reconnection were found. However, this new work studies flow-driven merging in the super-Alfvénic limit, with very strong flows ramming magnetic field together. In this situation, turbulence-induced X-points that might provide additional reconnection sites are usually rapidly ejected from the sheet. As discussed in the previous section, it is still a wide open question whether fast reconnection can occur in 2D MHD.

Some other numerical simulation papers have claimed to see evidence for fast reconnection in resistive MHD simulations. An important study of tearing instability of current sheets in the presence of background 2D turbulence and the formation of large-scale, long-lived magnetic islands has been performed in Politano et al. (1989). They present evidence for “fast energy dissipation” in 2D MHD turbulence and show that this result does not change as they change the resolution. A more recent work of Mininni & Pouquet (2009) provides evidence for “fast dissipation” also in 3D MHD turbulence. This phenomenon is consistent with the idea of fast reconnection, but cannot be treated as a direct evidence of the process. Evidently, fast energy dissipation and fast magnetic reconnection are rather different physical processes. A series of papers by Galsgaard and Nordlund, in particular Galsgaard & Nordlund (1997b), might also be interpreted as providing indirect evidence for fast reconnection. The authors noted that in their simulations they could not produce highly twisted magnetic fields. One of the interpretations of this finding could be the relaxation of magnetic field via reconnection. In this case, this observations could be related to the numerical finding of Lapenta & Bettarini (2011) which shows that reconnecting magnetic configurations spontaneously get chaotic and dissipate, which in its turn may be related to the predictions in LV99. However, in view of many uncertainties of the numerical studies, we are not confident of this connection. The highest resolution simulations of Galsgaard & Nordlund (1997b) were only 136^3 and at modest Lundquist numbers that could not permit a turbulent inertial-range.

A number of papers have attempted to treat turbulent magnetic reconnection by the formal methods of Mean-Field Electrodynamics (MFE), e.g. Strauss (1988); Kim & Diamond (2001). Before we address these works in any detail, let us briefly discuss the relationships of LV99 itself to MFE. LV99 theory as originally formulated does not consider ensemble- or time-averaged magnetic fields, does not employ MFE and, in fact, makes only very lim-

ited use of probability and averaging. The key observation to derive LV99 predictions using the exact formula (35) for the magnetic field—the “stochastic Lundquist formula”—is that it does *not* reduce to the conventional Lundquist formula as $\lambda \rightarrow 0$ in a turbulent plasma. The typical lateral spread of Lagrangian trajectories that contribute in that limit is estimated from GS95 turbulence theory to be $\ell_\perp^2 \sim \varepsilon |t - t_0|^3$ for $t < L_i/v_A$ and to be $\ell_\perp^2 \sim u_L L_i M_A^3 |t - t_0|$ for $t > L_i/v_A$. This is a turbulent mixing effect but not in the usual sense of an enhanced “eddy-diffusivity” or “ β -effect” experienced by a mean magnetic field. Within the latter concept (see Parker (1979)), magnetic fields are presumed to be mixed up passively by turbulence and the rate of reconnection thereby accelerated. This solution is, however, not tenable for any dynamical important magnetic field which would resist bending by turbulent motions at small scales. The concept in its original formulation is therefore not applicable to any astrophysical fields, apart from unrealistically weak fields, which are of marginal astrophysical importance, anyhow.

Of course, it is always possible to define in a purely formal manner an “effective diffusivity” which, when substituted into the MFE equations, will reproduce the predictions of *any* theory for the reconnection velocity v_{rec} and width Δ of the reconnection layer. All that one needs to do is set

$$\lambda_{turb} \equiv v_{rec} \Delta = \frac{L_x}{v_A} v_{rec}^2 \quad (70)$$

for the desired v_{rec} and for the corresponding layer width $\Delta \equiv L_x (v_{rec}/v_A)$ imposed by mass balance. In the case of LV99, this leads to a “turbulent magnetic diffusivity”

$$\lambda_{\perp turb} = u_L L_i M_A^3 \min \left\{ \left(\frac{L_x}{L_i} \right)^2, 1 \right\}, \quad (71)$$

where the subscript “ \perp ” denotes that this diffusivity is in the direction transverse to the background field. There is not necessarily any real physics associated with this purely formal definition of a diffusivity.

In the weak turbulence regime for $L_x > L_i$, however, this “turbulent magnetic diffusivity” reduces to the turbulent thermal diffusivity derived by Lazarian (2006). This is not an accident. As we have discussed in deriving eq.(60), the nearly straight field-lines in the weak turbulence regime experience a Brownian motion in the plane perpendicular to their direction, with a diffusion constant $\lambda_{\perp turb} = u_L L_i M_A^3$, as in (71). This implies a term in the induction equation for the mean field $\mathbf{B}(\mathbf{x}, t)$ of the form $\lambda_{\perp turb} \Delta_\perp \mathbf{B}$. There is therefore a consistent MFE approach to recover the predictions of the LV99 theory for reconnection in the weak turbulence regime, by means of an appropriate *anisotropic* turbulent magnetic diffusivity.

This is not the true for the strong turbulence regime with $L_x < L_i$. Recall from the discussion around eq.(60) that field lines in strong turbulence do *not* separate diffusively in the transverse direction, as $\ell_\perp^2(t) \sim \lambda_{\perp turb} t$, but instead undergo a Richardson diffusion $\ell_\perp^2(t) \sim \varepsilon t^3$. This superdiffusive line motion cannot be consistently described by a Laplacian term in the MFE equations. Note also that strict accuracy of the MFE description requires a slow variation of mean fields on the integral scale L_i

of the turbulence, whereas the width of the reconnection zone according to (62) always satisfies $\Delta < L_i$. One would therefore not expect MFE with a simple Laplacian diffusion term to consistently and accurately describe the reconnection layer in strong turbulence predicted by LV99.

With this discussion as backdrop, let us review some of the attempts to discuss magnetic reconnection using MFE, beginning with Kim & Diamond (2001) [KD01]. In the first place, it must be emphasized that they consider a rather unconventional set-up with a very strong guide field $\langle B_z \rangle$ and with fluctuating components B_{rms} also much larger than the reconnecting component $\langle B_H \rangle$, i.e. $\langle B_z \rangle \gg B_{rms} \gg \langle B_H \rangle$. Because of the first assumption, reconnecting lines of the mean field are almost exactly parallel, with only tiny components of opposing sign. Because of the second assumption $M_A \gg 1$ and there is no recognizable reconnection geometry for actual lines of the unaveraged magnetic field: the anti-parallel horizontal components are just a small mean effect. These conditions were adopted for analytical convenience and do not generally reflect astrophysical reality (particularly the second). Then KD01 develop closure approximations for the turbulent magnetic diffusivity using the 3D RMHD equations. Assuming that the small-scale fluctuations are statistically stationary and appealing to boundary conditions (e.g. periodic) which permit one to neglect surface flux terms, they derive a strongly quenched turbulent diffusivity with $\lambda_{turb} = O(\lambda)$, the plasma diffusivity. They therefore conclude that magnetic reconnection proceeds at the slow Sweet-Parker rate, in contradiction to LV99. However, we find their conclusions of little relevance to LV99. In the first place, they consider a quite different problem, with $M_A \gg 1$. In the second place, their main finding of a quenched turbulent magnetic diffusivity is due to the conservation of magnetic potential by the nonlinear terms of the RMHD equations. This quenching is unlikely to persist with the open boundary conditions required for stationary reconnection or to be a feature of turbulent reconnection governed by the full 3D MHD equations.

Finally, one should not mix up the concepts we discuss here with the so-called “hyper-resistivity” (Strauss 1986; Bhattacharjee & Hameiri 1986; Hameiri & Bhattacharjee 1987; Diamond & Malkov 2003), which is another attempt to derive fast reconnection from turbulence within the context of mean-field resistive MHD. The form of the parallel electric field can be derived from magnetic helicity conservation. Integrating by parts one obtains a term which looks like an effective resistivity proportional to the magnetic helicity current. There are several assumptions implicit in this derivation. Fundamental to the hyper-resistive approach is the assumption that the magnetic helicity of mean fields and of small-scale, statistically-stationary turbulent fields are separately conserved, up to tiny resistivity effects. However, this ignores magnetic helicity fluxes through open boundaries, essential for stationary reconnection, that vitiate the conservation constraint.

There is a general objection to all mean-field approaches which, to “explain” fast reconnection, make appeal to some effective dissipation (resistivity, hyper-resistivity, etc.) experienced by the fields once averaged over ensembles or over small space-time scales. The dif-

ficulty is that it is the lines of the *full* magnetic field that must be rapidly reconnected, not just the lines of the mean-field. The former implies the latter, but not conversely. Nature does not average over ensembles of small-scale turbulence to get fast reconnection! No mean-field approach can claim to have explained the observed rapid pace of magnetic reconnection unless it is shown that the reconnection rates obtained in the theory are strictly independent of the length- and time-scales of the averaging. See Eyink & Aluie (2006) for further discussion.

5.5. Turbulent Versus Collisionless Reconnection

The most popular alternative to LV99 is currently not a competing turbulence theory of reconnection but instead non-MHD approaches based on the Hall effect in a two-fluid model (Shay et al. 1998, 1999; Wang et al. 2000; Birn et al. 2001; Drake 2001; Malakit et al. 2009; Cassak et al. 2010), or on full kinetic Vlasov dynamics (Daughton et al. 2006, 2008, 2011; Che et al. 2011). We have argued in section 4.3 and at length in Appendix B that collisionless effects are, in most cases, irrelevant to determining reconnection speeds in the presence of turbulence. We do believe that there are some situations in certain astrophysical systems where kinetic effects will influence the rate of reconnection. However, formulating a criterion to determine if kinetic effects are important requires some care in the presence of turbulence. A popular criterion for “collisionless reconnection” is the condition that the Sweet-Parker thickness δ_{SP} be smaller than the ion-skin depth δ_i (Yamada et al. 2006; Zweibel & Yamada 2009). This condition has been checked to correctly signal “collisionless reconnection” in laminar simulations and experiments. However, as we now discuss, this condition in turbulent plasmas is not relevant to the validity of LV99.

There is even some question of how to define the “Sweet-Parker thickness” δ_{SP} in turbulent reconnection. In the original resistive MHD model of LV99, local, small-scale reconnection events were argued to have a Sweet-Parker Y-type structure and thickness

$$\ell_\eta^\perp \simeq L_i M_A^{-1} S_L^{-3/4}, \quad (72)$$

with ℓ_η^\perp the resistive cut-off length of GS95 turbulence theory. See LV99, eq.(14). However, this may not be the most relevant length-scale for determining the importance of collisionless effects. Within GS95 phenomenology one can estimate the pointwise ratio of the Hall electric field to the MHD motional field as

$$\frac{J/en}{u} \simeq \frac{c\delta B(\ell_\eta^\perp)/4\pi n e \ell_\eta^\perp}{u_L} \simeq \frac{\delta_i}{L_i} M_A S_L^{1/2} \quad (73)$$

where $S_L = v_A L_i / \lambda$ is the Lundquist number based on the forcing length-scale of the turbulence and $M_A = u_L / v_A$ is the Alfvénic Mach number. This suggests the definition of the “Sweet-Parker thickness” as

$$\delta_{SP} = L_i M_A^{-1} S_L^{-1/2}, \quad (74)$$

so that $(J/en)/u_L \simeq \delta_i / \delta_{SP}$. The length-scale δ_{SP} is the analogue of the Taylor microscale in hydrodynamic turbulence, whereas the distance ℓ_η^\perp is the GS95 analogue of the Kolmogorov scale. If the magnetic diffusivity in the definition of the Lundquist number is assumed to be that based on the Spitzer resistivity, given

by $\lambda = \delta_e^2 v_{th,e} / \ell_{ei}$ where δ_e is the electron skin depth, $v_{th,e}$ is the electron thermal velocity, and ℓ_{ei} is the electron mean-free-path length for collisions with ions, then $S_L = \left(\frac{m_e}{m_i}\right)^{1/2} \beta^{-1/2} \left(\frac{\ell_{ei}}{\delta_e}\right)^2 \left(\frac{L_i}{\ell_{ei}}\right)$, with $\beta = v_{th,i}^2 / v_A^2$ the plasma beta. Substituting into (73) and defining $M = v_{th,i} / u_*$ as the Mach number gives

$$\frac{\delta_i}{\delta_{SP}} \simeq \left(\frac{m_i}{m_e}\right)^{1/4} M \beta^{1/4} \left(\frac{\ell_{ei}}{L_i}\right)^{1/2}, \quad (75)$$

which coincides precisely with the ratio defined by Yamada et al. (2006), eq.(6).

With this definition of δ_{SP} , it is easy to see from the data in our Table 1 that the ratio $\delta_i / \delta_{SP} \simeq 10^{-3}$ for the warm ISM, $\simeq 1$ for post-CME current sheets and $\simeq 10^3$ for solar wind impinging on the magnetosphere. However, this has nothing to do with the validity of LV99 for those systems. What it does say is that Ohmic resistivity may not be the mechanism for line-breaking in the latter two cases and the original resistive model of LV99 is not an adequate description at sufficiently small scales. Thus, the structure of local, small-scale reconnection events should be strongly modified by Hall or other collisionless effects, with possibly an X-type structure, an ion layer thickness $\sim \delta_i$, quadrupolar magnetic fields, etc. However, these local effects are irrelevant to determining the global rate of large-scale reconnection (see Appendix B).

What is the correct criterion to determine a breakdown of LV99 and the importance of collisionless effects? The LV99 model assumes that the thickness Δ of the reconnection layer is set by turbulent MHD dynamics (line-wandering and Richardson diffusion). Thus, self-consistency requires that the length-scale Δ must lie within the range of scales where shear-Alfvén modes are correctly described by incompressible MHD. This implies a criterion for “turbulent collisionless reconnection”

$$\rho_i \gtrsim \Delta \quad (76)$$

with Δ calculated from eq.(62) and ρ_i the ion cyclotron radius (see section 2). Since $\Delta \propto L_x$, the large length-scale of the reconnecting flux structures, this criterion is far from being satisfied in most astrophysical settings. For example, in the three cases of Table 1, one finds using $\Delta = LM_A^2$ that $\rho_i / \Delta \simeq 10^{-13}$ for the warm ISM, $\simeq 10^{-6}$ for post-CME sheets, and $\simeq .1$ for the magnetosphere. Only in the latter case is the condition (76) close to being satisfied. Reconnection events in the magnetosphere probably do typically involve collisionless effects in an essential way. It must be appreciated, however, that this is a highly nongeneric situation and turbulent fluid effects will generally predominate in astrophysical reconnection.

Even in magnetospheric reconnection the presence of background turbulence in the environment should not be ignored. Like nearly every cosmic plasma, the magnetosphere commonly exhibits turbulence, either statistically steady or in sporadic bursts (Zimbardo et al. 2010). Energy spectra are generally similar to those in the solar wind, with spectral exponents close to -5/3 and -7/3 at scales above and below the ion gyroradius/ion skin-depth (since $\beta \simeq 0.1 - 10$, these are nearly the same), respectively. See Table 1 in Zimbardo et al. (2010). The origi-

nal LV99 theory does not apply in the situation that $\rho_i \simeq \Delta > \rho_e$, but a version of LV99 based on EMHD might provide some insight. Of course, since ρ_e is only 43 times smaller than ρ_i for a hydrogen plasma, any EMHD inertial range must be of limited extent. Kinetic effects enter at electron scales, not accurately described by a fluid model, and approaches based on solving the Vlasov equation (see section 6.3) become important. Some insights of LV99 should still carry over, such as the importance of field-line wandering in enhancing reconnection rates. See section 6.3 for more discussion.

5.6. Observational and Numerical Tests

Let us discuss briefly the support for LV99 from observations and simulations.

The solar corona is a system proximate to Earth where LV99 theory should be applicable. Recent observations of a greatly thickened current sheet associated with coronal mass ejections give broad support to LV99 (Ciaravella & Raymond 2008; Bemporad et al. 2008). For example, the width Δ of the reconnection zone from the formula (62) was found by Ciaravella & Raymond (2008) to be just 10 times smaller than the observed thickness. This is quite good agreement, given that that formula has an unknown prefactor (which probably depends upon global geometry) and that some of the inputs (particularly the turbulence integral scale L_i) were unknown from observations and had to be crudely estimated. LV99 predicted also the phenomenon of triggering of magnetic reconnection. Indeed, as the reconnection speed depends on the level of magnetic field stochasticity, the stochasticity induced by Alfvén wave packages can enhance the reconnection speed. This is consistent with the recent observations by Sych et al. (2009) who reported a phenomenological relationship between oscillations in a sunspot and flaring energy release above an active region above the sunspot. The authors proposed that the pulsations in the flaring energy release are triggered by wave packages arising from sunspots. The LV99 mechanism presents a very natural explanation for the phenomenon.

While the Sun presents one of the best studied examples of magnetic reconnection, interstellar medium has the advantage of testing LV99 in a collisional environment (see Yamada (2007)). Similarly, collisional reconnection was observed in the solar photosphere (Park et al. 2009). For both types of environments effects of partial ionization may be important¹⁴. The generalization of the LV99 model for a partially ionized gas is presented in Lazarian et al. (2004). The direct numerical testing of the regimes of reconnection described in that paper requires the use of a two-fluid code and has not been done so far. Given the importance of magnetic fields in interstellar medium, such a testing is very appealing.

The most convincing tests of LV99 would be of its predicted scaling laws for the width (62) and for the reconnection velocity (64) in terms of the parameters u_L , v_A , L_i , and L_x . These scalings are most easily checked in numerical MHD studies, which achieve only moderate Reynolds numbers but which permit controlled experiments. Kowal et al. (2009) used the relation $\varepsilon \simeq u_L^4 / v_A L_i$

¹⁴ For the interstellar media some phases are partially ionized (see Draine & Lazarian (1998) for a list of idealized phases and their parameters).

to rewrite formula (64) as

$$v_{\text{rec}} = \left(\frac{\varepsilon}{v_A L_x} \right)^{1/2} \min\{L_x, L_i\}. \quad (77)$$

and tested the scalings in ε and L_i by means of numerical simulations. Kowal et al. (2009) studied in particular the weak-turbulence regime with $L_x > L_i$, investigating the predicted power-law scalings $v_{\text{rec}} \propto \varepsilon^{1/2} L_i$. The dependence on $\varepsilon^{1/2}$ was confirmed, but the predicted linear relation $\propto L_i$ was not well verified and replaced with a weaker dependence closer to $L_i^{3/4}$ for larger $L_i \lesssim L_x$. This behavior is possibly associated with a crossover to the strong-turbulence regime with $L_i > L_x$. Note that the formula (77) for strong turbulence predicts independence of the reconnection rate from L_i , except through the dependence on L_i of the mean energy dissipation ε .

The independence of reconnection rate on L_i for strong turbulence is quite convenient for observational tests of LV99, since L_i is much harder to reliably estimate in natural reconnection phenomena than are v_A , L_x and ε . Indeed, the source of turbulence may not be so easily identified or localized in wavenumber. One approach is to obtain u_L from Doppler line-broadening (e.g. Bemporad et al. (2008)) and then to estimate $L_i = u_L^4 / v_A \varepsilon$. Of course, if spectral information is available, then one can obtain L_i directly from the peak of the energy spectrum. The mean energy dissipation rate ε is a source term for plasma heating, which can be estimated from observations of electromagnetic radiation. To make such an estimation reliably requires not only a radiation model but also some understanding of where the energy from the cascade is deposited at scales smaller than the ion gyro-radius (Schekochihin et al. 2009). It may be preferable to estimate ε from coarse-grained measurements of the energy-flux rate Π_ℓ , which is equal, on average, to the dissipation rate. For example, the flux of magnetic energy is given by $\Pi_\ell^B = \bar{\mathbf{J}}_\ell \cdot \boldsymbol{\varepsilon}_\ell$ where $\bar{\mathbf{J}}_\ell = (c/4\pi) \nabla \times \bar{\mathbf{B}}_\ell$ is the coarse-grained current and $\boldsymbol{\varepsilon}_\ell = (\mathbf{u} \times \bar{\mathbf{B}})_\ell - \bar{\mathbf{u}}_\ell \times \bar{\mathbf{B}}_\ell$ is the subscale EMF. Because of the scale-locality of the MHD energy cascade it is possible to develop very quantitatively accurate approximations to the subscale EMF which depend only upon the coarse-grained gradients $\nabla \bar{\mathbf{u}}_\ell$ and $\nabla \bar{\mathbf{B}}_\ell$. For discussion of the approximation, see Eyink (2006).

Additional testing of magnetic reconnection can be produced through indirect means. Indeed, the process of reconnection may be responsible for a wide variety of processes. For instance, the LV99 model of reconnection was invoked to explain gamma ray bursts (Lazarian et al. 2002; Zhang & Yan 2011), acceleration of cosmic rays (de Gouveia Dal Pino & Lazarian 2003; Lazarian 2005; Lazarian & Opher 2009; Lazarian & Desiati 2010), removal of magnetic field during star formation (Lazarian 2005; Santos-Lima et al. 2010). As the processes are quantified and elaborated, it is possible to get insight into the magnetic reconnection that is driving them.

6. DISCUSSION

Let us discuss here briefly the main results of this paper as well as some of their ramifications.

6.1. Flux-freezing in astrophysics

Plasma conductivity is high for most astrophysical circumstances. It has therefore generally been assumed that “flux-freezing” is an excellent approximation for astrophysical plasmas. The principle of “frozen-in” field-lines provides a powerful heuristic which allows simple, back-of-the-envelope estimates in place of full solutions (analytical or numerical) of the MHD equations. As such, the “flux-freezing” principle has been applied to gain insight into diverse processes, such as star formation, stellar collapse, magnetic dynamo, solar wind-magnetospheric interactions, etc. The predictions of this simple principle often accord very well with observations. Flux-freezing is used to explain, for example, the magnetization of white dwarfs and neutron stars, the low angular momentum of stars, and the spiral structure of lines of force in the solar wind.

However, for every success of the “flux-freezing” principle, there is an equally stark failure. If the standard “flux-frozenness” property held to good accuracy, then topology changes of magnetic field could not occur at the very fast rates observed in solar flares and coronal mass ejections. During star-formation, the magnetic pressure of in-falling field-lines would be so great as to prevent gravitational collapse altogether. The tangled line-structure in small-scale dynamos would quench the exponential growth of magnetic field. The most common attempts to explain such contrary observations invoke additional physical effects that violate conventional “flux-freezing”, e.g. additional terms in the generalized Ohm’s law besides collisional resistivity. Unfortunately, most of these microscopic plasma effects seem to be too small to have much effect in astrophysical MHD plasmas at very high kinetic and magnetic Reynolds numbers.

Turbulence is a natural suspect to explain apparent breakdowns of “flux-freezing” in astrophysical environments. Most attempts to use turbulence invoke it as just another effective non-ideal term in the generalized Ohm’s law. Indeed, any averaging of Ohm’s law (e.g. over ensembles, space or time) produces an additional “turbulent EMF” that leads to a possible resistivity-independent violation of “flux-freezing” for the *mean* magnetic field. This is a correct observation as far as it goes, but not a fundamental solution to the problem. “Flux-frozenness” in the conventional sense must be violated for the full magnetic field and not just for some averaged field.

Our solution is quite different. We have argued that “flux-freezing” in the standard sense is violated in a turbulent inertial range at high kinetic and magnetic Reynolds numbers, but continues to hold in a novel stochastic sense. This *stochastic flux-freezing law* in turbulent plasmas is a consequence of the remarkable turbulence phenomenon of “spontaneous stochasticity” of Lagrangian particle trajectories. As we have shown in detail, it is this stochastic form of “flux-freezing” which underlies the LV99 theory of fast magnetic reconnection. In any turbulent astrophysical system, “stochastic flux-freezing” is the natural replacement for conventional flux freezing. We believe that it will provide a powerful heuristic tool to explain many astrophysical phenomena. To use this tool with confidence requires a sound understanding of MHD plasma turbulence, in particular Lagrangian particle statistics. This is a clear focus of further theoretical, experimental and numerical work.

6.2. Line-wandering and LV99

Spontaneous stochasticity is a property not only of Lagrangian trajectories, but also of magnetic field-lines themselves. The identity of “the” field-line passing through a point becomes blurred in a high magnetic-Reynolds-number turbulent plasma. Certainly, at any moment in time there are a unique set of field lines that describe the field, through each point. However, there are an infinite number of field lines that can be drawn through a small volume, and if we follow for some parallel distance along these field lines, then they diverge explosively away from one another. In a turbulent plasma with a rough magnetic field, the distance that these lines diverge apart is independent of the dimensions of this initial volume, when the parallel distance travelled greatly exceeds this dimension. As a limiting situation, for vanishingly small initial volumes, there are infinitely many field-lines that emerge and meander randomly out of one point! At sufficiently large scales, that is, above the dissipation scale, this line-wandering is independent of the value of resistivity, and, within limits, to the details of the microscopic physics (see, for example, Appendix B).

This was the original insight of Lazarian & Vishniac (1999). The important implication is that plasma, flowing a finite distance along magnetic fields lines, will diffuse away from the mean magnetic field direction. This effect can accelerate reconnection speeds not only in the presence of inertial-range turbulence, but whenever field-lines exhibit some disordered, random character. If we use line-wandering in a turbulent reconnection zone to calculate the width of the outflow region around the current sheet, then we find that the classic Sweet-Parker rate becomes the fast reconnection rate given in LV99, independent of resistivity. As we have shown in the present paper, this same reconnection rate can also be obtained by an alternative approach in which an ensemble of field lines move stochastically in time. This agreement follows from a formal equivalence between the two approaches.

The precise quantitative details of magnetic line-wandering depend upon the scaling laws of MHD turbulence. We have adopted here the predictions of phenomenological GS95 theory, and these should surely be verified and possibly refined in future numerical and experimental studies. However, a parameter study in LV99 shows that the conclusion of fast reconnection in a 3D turbulent flow holds for a range of spectral slopes and spectrum anisotropies that encompass the existing suggestions of modifying the GS95 model. Our work neglects also some features of large scale reconnection. By treating the reconnection zone as a *typical* part of a turbulent plasma we are neglecting the effects of a strongly sheared large scale field and the fast outflow along magnetic fields that will accompany reconnection events. These features are naturally included in numerical simulations (Kowal et al. 2009) and do not seem to result in any significant modification of the reconnection rate, but further work along this line is certainly desirable.

6.3. Other recent work on reconnection

Since our conclusions are consistent with the work of LV99 it is worth considering recent work on alternative approaches to calculating reconnection rates. Over the last decade, more traditional approaches to reconnection

have changed considerably. At the time of its introduction, the models competing with LV99 were modifications of the single X-point collisionless reconnection scheme first introduced by Petschek (1964). Those models had point-wise localized reconnection regions which were stabilized via plasma effects so that the outflow opened up on larger scales (see Fig. 5). Such configurations would be difficult to realize in the presence of random forcing, which would be expected to collapse the reconnection layer. Moreover, Ciaravella & Raymond (2008) argued that observations of solar flares were inconsistent with single X-point reconnection.

In response to these objections, more recent models of collisionless reconnection have acquired several features in common with the LV99 model. In particular, they have moved to consideration of volume-filling reconnection, (although it is not clear how this volume filling is achieved in the presence of a single reconnection layer (see Drake et al. (2006))). While much of the discussion still centers around magnetic islands produced by reconnection, in three dimensions these islands are expected to evolve into contracting 3D loops or ropes due to tearing-type instabilities in electron-current layers (Daughton et al. (2008, 2011)). This is broadly similar to what is depicted in Fig. 5, at least in the sense of introducing stochasticity to the reconnection zone. The 3D PIC simulation studies reported in these works should help, in particular, to interpret magnetospheric reconnection phenomena. However, although the simulations are described as “turbulent”, they do not exhibit the inertial-range power-law spectra observed in the magnetosphere and do not take into account either the pre-existing turbulence found in many of its regions (due to temperature anisotropy, velocity shear, Kelvin-Helmholtz instability, etc.) or inertial-range turbulence generated as a consequence of reconnection itself (Zimbardo et al. 2010). An EMHD analogue of LV99 and Kowal et al. (2009) studies may help to estimate such effects.

Similar remarks apply to the recent 3D PIC study of Che et al. (2011), which observes micro-turbulence in the electron current layer during reconnection. The authors identify the source of this “turbulence” as a filamentation instability driven by current gradients, very similar to a related instability in the EMHD model. The key aim of this work was to identify the term in the generalized Ohm’s law which supplies the reconnection electric field to break the “frozen-in” condition. However, this study ignores the ambient inertial-range turbulence observed in the magnetosphere and other astrophysical plasmas, which may strongly modify laminar instabilities. Also, while there is interest in understanding the origin and character of reconnection electric fields (e.g. for particle acceleration), we have argued at length in section 4.3 that the precise mechanism of “line-breaking” is irrelevant for the rate of reconnection in the presence of high-Reynolds-number inertial-range turbulence.

Departure from the concept of laminar reconnection and the introduction of magnetic stochasticity is also apparent in a number of the recent papers appealing to the tearing mode instability to drive fast MHD reconnection (Loureiro et al. 2009; Bhattacharjee et al. 2009).¹⁵

¹⁵ The idea of appealing to the tearing mode as a means of enhancing the reconnection speed can be traced back to Strauss

Several 2D numerical studies (Samtaney et al. 2009; Cassak et al. 2009; Bhattacharjee et al. 2009; Huang & Bhattacharjee 2010) have provided evidence that reconnection in the plasmoid-unstable region is independent of resistivity and a simple, heuristic picture of a multi-level plasmoid hierarchy has been proposed for this regime by Uzdensky et al. (2010). A recent very high-resolution study of Ng & Raghunathan (2011), however, has found that plasmoids do not form even when the Lundquist number exceeds the putative critical value, except when insufficient numerical resolution is used or when a small amount of noise is added externally. Thus, this alternative MHD mechanism of fast reconnection may require some level of environmental noise, similar to LV99. In any case, since tearing modes exist even in a collisional fluid, this may open another channel of reconnection in such fluids. As we discuss below, this reconnection should not be “too fast” to account for the observational data.

A fundamental consideration for all reconnection models is that they must explain fast reconnection in collisional and collisionless plasmas. At the same time it should be possible for reconnection to be delayed for significant amounts of time. Otherwise the accumulation of magnetic flux prior to a solar flare would be inexplicable. Fast reconnection as a feature of turbulence sets a minimum reconnection speed which allows the topology of the magnetic field to evolve on dynamical time scales. Thus the LV99 model can explain the accumulation of flux provided that there are no generically fast reconnection processes at work¹⁶. An alternative explanation based on collisionless reconnection in a laminar environment has been suggested by Cassak et al. (2006) and Uzdensky (2007). However, it is not clear whether this explanation will account also for the large observed thickness of the macroscopic current sheets (Ciaravella & Raymond 2008; Bemporad et al. 2008).

If local turbulence is driven by release of energy from the magnetic field, it may result in a runaway turbulent reconnection process which may be relevant to some numerical simulations (Lapenta 2008; Bettarini & Lapenta 2009). Alternatively, if tearing modes begin by driving relatively slow reconnection, but by some process destabilize the current sheet, then a similar runaway might result. Turbulence looks like a natural candidate for such process, but one should be open to alternative suggestions. For instance, we note that while X-point reconnection is clearly unstable for an isolated current sheet in a collisional fluid, interacting current sheets can produce bursts of fast X-point reconnection, separated by periods of slow evolution (Pang, Pen, & Vishniac (2010)). For-

(1988); Waelbroeck (1989); Shibata & Tanuma (2001). LV99 showed that the linear growth of tearing modes is insufficient to obtain fast reconnection. More recent work is based on the idea that the non-linear growth of magnetic islands or plasmoids due to mergers provides large scale growth rates larger than the tearing mode linear growth rates on these scales. A situation where the non-linear growth is faster than the linear one is rather unusual and requires further investigation. See Diamond et al. (1984).

¹⁶ To be specific, take typical coronal loop parameters $n = 10^9 \text{ cm}^{-3}$, $T = 100 \text{ eV}$, $B = 300 \text{ G}$, $u_L = 10^7 \text{ cm/s}$ and $L_i \simeq L_x = 10^9 \text{ cm}$. Then $v_A = 10^{10} \text{ cm/s}$ and $\Delta = L_i M_A^2 = 10^3 \text{ cm}$, comparable to the ion skin depth $\delta_i = 700 \text{ cm}$. Thus, the initial turbulent reconnection rate will be quite small and may be slower than collisionless mechanisms. However, the energy released in the slow reconnection process can make the region more turbulent, accelerating the reconnection and resulting in a flare.

mally, in this case we might not need to invoke turbulent feedback¹⁷, but simply rely on the relatively slow pace of stochastic reconnection to evolve the magnetic field from one outburst to the next.

In any case, in most astrophysical situations one has to deal with the *pre-existing turbulence*, which is the inevitable consequence of the high Reynolds number of astrophysical fluids and for which abundant empirical evidence exists. Such turbulence may modify or suppress instabilities, including the tearing mode instability. In this paper we have shown that it, by itself, induces fast reconnection on dynamical time scales.

6.4. Outstanding Issues for Turbulent Reconnection

Much remains to be done to fully clarify the effects of MHD plasma turbulence on astrophysical reconnection. Without attempting to be exhaustive, let us mention some of the issues that we regard as most pressing.

Of the possible extensions and refinements of LV99 mentioned in section 5, we think one of the most important is including the effects of magnetic and velocity shear. A related issue not addressed in the current paper is the detailed structure of the turbulent reconnection zone. We have invoked here the GS95 phenomenology of MHD turbulence, but intermittency effects not included in GS95 could be relevant. In principle, an independent theoretical derivation of the reconnection speed can be based on the line-voltage studied in Kowal et al. (2009). The arguments of LV99 suggest that the reconnection voltage should be contributed primarily by the motional EMF of already reconnected field-lines, but a detailed analysis is required. Addressing these issues will be the subject of future work.

In its original formulation, LV99 is a model of non-relativistic reconnection. However, it is clear that the idea of enabling fast reconnection by extending the thickness of the outflow region through magnetic field wandering should be applicable to relativistic flows. This extension was implicitly used in some models of gamma ray bursts (see Lazarian et al. (2002); Zhang & Yan (2011)). Formal and detailed studies of relativistic turbulent reconnection are therefore important.

Similarly, inertial-range turbulence with power-law spectra is an idealization adopted in LV99 to get analytically tractable results. In many cases, however, wandering of field-lines and of Lagrangian trajectories is significantly affected by larger or smaller scales outside the inertial-range. Deviations from pure power-law spectra may occur due to multiple turbulent energy injection scales or to the effects of Ohmic dissipation. These effects of inhomogeneity of turbulence driving are important subjects for further quantitative studies.

Last, but not least, effects of turbulence dissipation via viscosity is another issue where more studies are required. Lazarian et al. (2004) extended LV99 to the case of partially ionized gas where the dissipation of turbulence arises mostly through viscosity. The concept of Richardson diffusion and its relation to turbulent reconnection in this case needs additional work.

¹⁷ Energy release induces the feedback anyhow and the interaction of the resulting turbulence with the instabilities is an issue of further research.

7. SUMMARY

The main points of our paper can be very briefly summarized as follows:

1. In turbulent plasmas with rough velocity fields, the lines of force are not “frozen-in” in the standard deterministic sense, but instead in a random sense associated to the “spontaneous stochasticity” of Lagrangian particle trajectories.
2. Magnetic reconnection rates calculated based on “stochastic flux-freezing” and phenomenological GS95 turbulence theory recover the predictions of the LV99 theory of turbulent reconnection.
3. The LV99 predictions are independent of all non-ideal terms in the generalized Ohm’s law. For

example, they do not change if the Hall effect is included, whenever the ion skin depth is much smaller than the turbulent integral length or injection scale.

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APPENDIX

FORMAL TREATMENT OF TURBULENT LINE-SEPARATION

Consider the problem of the growth in separation of a pair of magnetic field lines, starting at points displaced by vector ℓ , as one moves a distance s in arclength along the field lines passing through the two points (cf. Jokipii (1973)). The exact equation for the change in separation can be obtained from (45) to be

$$\frac{d}{ds}\ell(s) = \hat{\mathbf{b}}(\xi'(s)) - \hat{\mathbf{b}}(\xi(s)), \quad (\text{A1})$$

where s is arclength, $\ell(s) = \xi'(s) - \xi(s)$, and $\hat{\mathbf{b}} = \mathbf{B}/|\mathbf{B}|$ is the unit tangent vector along the magnetic field-line. By analogy with Richardson diffusion for Lagrangian particle trajectories, one can define a “2-line diffusivity”

$$D_{ij}^B(\ell) = \int_{-\infty}^0 ds \langle \delta \hat{b}_i(\ell, 0) \delta \hat{b}_j(\ell, s) \rangle, \quad (\text{A2})$$

with units of length, where $\delta \hat{b}_i(\ell, s) = \hat{b}_i(\xi'(s)) - \hat{b}_i(\xi(s))$ with $\ell = \xi'(0) - \xi(0)$, so that

$$\frac{d}{ds} \langle \ell_i(s) \ell_j(s) \rangle = \langle D_{ij}^B(\ell) \rangle. \quad (\text{A3})$$

The above equations are formally exact (Batchelor 1950; Kraichnan 1966). Note that (A2) allows one to write

$$D_{ij}^B(\ell) \sim \delta \hat{b}_i(\ell) \delta \hat{b}_j(\ell) s_{int}(\ell) \quad (\text{A4})$$

where $s_{int}(\ell)$ is an integral correlation length of the increment in the tangent vector moving along the lines. (This is properly a tensor quantity, but here written as a scalar.) To determine the line wandering, one must have a model for this integral correlation length.

The LV99 theory is obtained by modelling the integrand in (A2) for field-perpendicular increments as

$$\langle \delta \hat{b}_\perp(\ell, 0) \delta \hat{b}_\perp(\ell, s) \rangle \sim \frac{\delta u^2(\ell)}{v_A^2} \text{Re}(e^{is/\ell_\parallel - |s|/\lambda(\ell)}). \quad (\text{A5})$$

Note that $|\delta \hat{b}_\perp(\ell)| \sim \delta B(\ell)/B_0 \sim \delta u(\ell)/v_A$. There are two effects in the s -dependent part. First, there is a correlation length $\lambda(\ell)$ of tangent-vector increments along the field line, which gives an exponential decay. Within LV99 theory, it is assumed that this length is the distance traveled by a random Alfvénic disturbance propagating along the field-line with velocity v_A in one energy cascade time at scale ℓ_\perp . That is,

$$\lambda(\ell) = v_A \tau_\ell = v_A \frac{\delta u^2(\ell)}{\varepsilon}. \quad (\text{A6})$$

The second effect arises from the periodic variation along the field-lines due to regular Alfvén wave trains. It is assumed that, at perpendicular scale ℓ_\perp , the Alfvén wave trains that are dominant are those with wavelength ℓ_\parallel , the parallel length-scale corresponding to perpendicular scale ℓ_\perp . Integrating (A5) in s gives the result

$$s_{int}(\ell) \sim \frac{1/\lambda(\ell)}{1/\lambda^2(\ell) + 1/\ell_\parallel^2} \sim \frac{\ell_\parallel^2}{\lambda(\ell)} = \frac{\varepsilon}{v_A} \frac{\ell_\parallel^2}{\delta u^2(\ell)} \quad (\text{A7})$$

for $\lambda(\ell) \geq \ell_{\parallel}$. Substituting into (A4) gives

$$D_{\perp}^B(\ell) \sim \frac{\varepsilon \ell_{\parallel}^2}{v_A^3} = \frac{\ell_{\parallel}^2}{L_i} M_A^4, \quad (\text{A8})$$

where notice that a factor of $\delta u^2(\ell)$ has cancelled between numerator and denominator. From this last formula, various specific cases can be considered.

In the strong GS95 turbulence regime, critical balance implies that $\lambda(\ell) \sim \ell_{\parallel}$ and thus, from (A7), $s_{int}(\ell) \sim \ell_{\parallel}$. This gives the intuitive result that

$$D_{\perp}^B(\ell) \sim (\delta u_{\ell}/v_A)^2 \ell_{\parallel} \sim L_i \left(\frac{\ell_{\perp}}{L_i} \right)^{4/3} M_A^{4/3}, \quad (\text{A9})$$

where we have substituted from (5),(6) for ℓ_{\parallel} and $\delta u_{\ell}/v_A$. Alternatively, one can just substitute for ℓ_{\parallel} from (5) into (A8). The equation

$$\frac{d}{ds} \ell_{\perp}^2 \sim D_{\perp}^B(\ell) \sim L_i \left(\frac{\ell_{\perp}}{L_i} \right)^{4/3} M_A^{4/3}, \quad (\text{A10})$$

is then easily solved to give eq.(8) in the text for $L_i > s \gg \ell_{\perp}^{(0)}$. This result was already obtained for $M_A = 1$ by Skilling et al. (1974) in a discussion of cosmic-ray diffusion, assuming a simple isotropic Kolmogorov hydrodynamic model of magnetized turbulence. In the weak turbulence regime one has instead $\ell_{\parallel} = L_i$ a constant, which substituted into (A8) gives

$$\frac{d}{ds} \ell_{\perp}^2 \sim D_{\perp}^B(\ell) \sim L_i M_A^4 \quad (\text{A11})$$

and this is solved to give the diffusive result (9) in the text.

NEGLECT OF THE HALL TERM IN TURBULENT RECONNECTION

All of the nonideal terms in the generalized Ohm's law (47)—Ohmic resistivity, Hall term, pressure tensor, electron inertia, etc.—are expected to be insignificant in a long turbulent inertial range, at sufficiently large length-scales. Thus, they do not alter the predictions of the LV99 theory for the reconnection rates of large-scale flux structures. We shall here present more detailed analytical arguments for this thesis, in the specific example of the incompressible Hall MHD (HMHD) equations. As remarked in the text, the HMHD dynamics is valid in a literal sense for almost no astrophysical plasmas. However, it has been used in many reconnection studies as a simple fluid model that exemplifies the Hall effect (Shay et al. 1998, 1999; Wang et al. 2000; Birn et al. 2001; Drake 2001; Malakit et al. 2009; Cassak et al. 2010), so that it is a useful example to consider in this respect.

The incompressible Hall MHD equations can be written as

$$(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \mathbf{j} \times \mathbf{b} + \nu \Delta \mathbf{u}, \quad (\text{B1})$$

$$\partial_t \mathbf{b} = \nabla \times [(\mathbf{u} - \delta_i \mathbf{j}) \times \mathbf{b}] + \lambda \Delta \mathbf{b}, \quad (\text{B2})$$

with $\nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{b} = 0$ and $\mathbf{j} = \nabla \times \mathbf{b}$. Here we have introduced the Alfvén variable $\mathbf{b} = \mathbf{B}/\sqrt{4\pi\rho}$ with units of velocity and $\delta_i = c/\omega_{p,i}$ is the ion skin depth. The above system has two inviscid constants of motion that cascade to small scales, the total energy

$$E = \frac{1}{2} \int d^3x [|\mathbf{u}(\mathbf{x})|^2 + |\mathbf{b}(\mathbf{x})|^2] \quad (\text{B3})$$

and the cross-helicity

$$H_C = \int d^3x \mathbf{u}(\mathbf{x}) \cdot [\mathbf{b}(\mathbf{x}) + \frac{1}{2} \delta_i \nabla \times \mathbf{u}(\mathbf{x})], \quad (\text{B4})$$

as well as the magnetic helicity which cascades to large scales. It is not our purpose here to give an exhaustive account of the turbulent cascade phenomenology of the HMHD model. Instead, we just wish to establish the simple fact that the Hall term modifies turbulence properties only at length-scales $\ell \lesssim \delta_i$, whereas length-scales $\ell \gtrsim \delta_i$ will behave as in standard MHD turbulence without the Hall term.

Our first argument is based on a spatial coarse-graining approach in an Eulerian formulation (e.g. see Eyink & Aluie (2006)). The HMHD induction equation coarse-grained (low-pass filtered) at length-scale ℓ becomes

$$\partial_t \bar{\mathbf{b}}_{\ell} = \nabla \times [(\bar{\mathbf{u}}_{\ell} - \delta_i \bar{\mathbf{j}}_{\ell}) \times \bar{\mathbf{b}}_{\ell} + \varepsilon_{\ell} - \lambda \bar{\mathbf{j}}_{\ell}], \quad (\text{B5})$$

where the total contribution to the turbulent subscale EMF is

$$\varepsilon_{\ell} = \overline{(\mathbf{u}_e \times \mathbf{b})}_{\ell} - \bar{\mathbf{u}}_{e\ell} \times \bar{\mathbf{b}}_{\ell}. \quad (\text{B6})$$

and $\mathbf{u}_e \equiv \mathbf{u} - \delta_i \mathbf{j}$ is the electron fluid velocity. It is easy to see that

$$\frac{\delta_i \bar{\mathbf{j}}_{\ell}}{\bar{\mathbf{u}}_{\ell}} \simeq \frac{\delta_i \delta b(\ell)/\ell}{u_L} \simeq \frac{\delta_i}{\ell} M_A^{1/3} \left(\frac{\ell}{L_i} \right)^{1/3}, \quad (\text{B7})$$

where the last expression uses GS95 theory to obtain the $1/3$ powers of M_A and ℓ/L_i . However, a similar expression will hold in any theory of HMHD turbulence in which $\delta b(\ell) \sim v_A M_A^q (\ell/L_i)^p$ with $q \geq 1$ and $p > 0$. Hence $\delta_i \bar{j}_\ell \ll \bar{u}_\ell$ for $\delta_i \ll \ell \ll L_i$, so that the coarse-grained dynamics in HMHD turbulence at those scales is identical in form to standard MHD. The only difference is that the EMF (B6) has an additional contribution from the Hall term

$$\boldsymbol{\varepsilon}_\ell^{Hall} = -\delta_i [\overline{(\mathbf{j} \times \mathbf{b})}_\ell - \bar{\mathbf{j}}_\ell \times \bar{\mathbf{b}}_\ell] = \frac{1}{2} \delta_i \boldsymbol{\nabla} [\text{tr}(\boldsymbol{\tau}_{M\ell})] - \delta_i \boldsymbol{\nabla} \cdot \boldsymbol{\tau}_{M\ell}, \quad (\text{B8})$$

where $\boldsymbol{\tau}_{M\ell} = \overline{(\mathbf{b}\mathbf{b})}_\ell - \bar{\mathbf{b}}_\ell \bar{\mathbf{b}}_\ell$ is the turbulent Maxwell stress. (Notice incidentally that the gradient term in (B8) does not contribute at all when substituted into the curl in (B5).) The total EMF in (B6) is thus $\boldsymbol{\varepsilon}_\ell = \boldsymbol{\varepsilon}_\ell^{MHD} + \boldsymbol{\varepsilon}_\ell^{Hall}$ including the standard MHD contribution $\boldsymbol{\varepsilon}_\ell^{MHD} = \overline{(\mathbf{u} \times \mathbf{b})}_\ell - \bar{\mathbf{u}}_\ell \times \bar{\mathbf{b}}_\ell$. It is easy to estimate the size of these various terms as

$$\boldsymbol{\varepsilon}_\ell^{MHD} \sim \delta u(\ell) \delta b(\ell), \quad \boldsymbol{\varepsilon}_\ell^{Hall} \sim \delta_i \frac{\delta b^2(\ell)}{\ell}, \quad \bar{\mathbf{E}}_\ell^{Ohm} = \lambda \bar{\mathbf{j}}_\ell \sim \lambda \frac{\delta b(\ell)}{\ell} \quad (\text{B9})$$

These estimates are rigorous upper bounds (cf. Eyink (2005)) but they must be good order of magnitude estimates of the terms as well, in order to allow constant fluxes of energy and cross helicity to small scales. In that case we must interpret ℓ in the upper bounds as ℓ_\perp , the scale perpendicular to the mean magnetic field¹⁸ Instead at scales $\ell < \delta_i$ the terms proportional to δ_i dominate, so that

$$\Pi_\ell^E \sim -\bar{\mathbf{j}}_\ell \cdot \boldsymbol{\varepsilon}_\ell^{Hall} \sim \delta_i \frac{\delta b^3(\ell)}{\ell_\perp^2}, \quad \Pi_\ell^C \sim -\delta_i \boldsymbol{\nabla} \bar{\boldsymbol{\omega}}_\ell : \boldsymbol{\tau}_\ell - \bar{\boldsymbol{\omega}}_\ell \cdot \boldsymbol{\varepsilon}_\ell^{Hall} \sim \delta_i \frac{\delta u(\ell)}{\ell_\perp^2} [\delta u^2(\ell) + \delta b^2(\ell)].$$

One thus gets constant fluxes with the scaling

$$\delta u(\ell) \sim \delta b(\ell) \sim (\varepsilon \ell_\perp^2 / \delta_i)^{1/3},$$

in agreement with the numerical results of Shaikh & Shukla (2009). As a matter of fact, the precise scaling at length-scales $\ell < \delta_i$ will not affect our conclusions. As usual, the large-scale Ohmic electric field is negligible compared with the MHD turbulent EMF whenever $\ell \delta u(\ell) \gtrsim \lambda$, or $\ell \gtrsim \ell_\eta^\perp = (\lambda^3 / \varepsilon)^{1/4}$ assuming GS95¹⁹. However, one can also see that $\boldsymbol{\varepsilon}_\ell^{Hall}$ is negligible compared with $\boldsymbol{\varepsilon}_\ell^{MHD}$ whenever $\ell \gtrsim \delta_i$. Here we have assumed that $\delta b(\ell) \simeq \delta u(\ell)$, which is true in GS95 but also in any theory of MHD turbulence which predicts equipartition of kinetic and magnetic energies at small scales. These considerations show that the coarse-grained dynamics in Hall MHD turbulence reduce to those in standard MHD turbulence at length-scales $\ell \gtrsim \delta_i$. This conclusion is in agreement with the numerical study of Dmitruk & Matthaeus (2006). They compared results of Hall and standard MHD simulations at kinetic and magnetic Reynolds numbers of 1000, with δ_i for the Hall simulation chosen about 4 times greater than the resistive dissipation scale. They found virtually identical results in the two simulations at scales greater than δ_i for all of the various quantities, such as velocities, magnetic fields and electric fields.

Now let us consider the Lagrangian description of HMHD turbulence. Stochastic flux-freezing holds also in resistive Hall MHD, with magnetic field-lines stochastically frozen-in to the electron fluid velocity \mathbf{u}_e (Eyink 2009). Thus, the lines can be regarded to “move” according to the stochastic equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{x}, t) - \delta_i \mathbf{j}(\mathbf{x}, t) + \sqrt{2\lambda} \boldsymbol{\eta}(t) \quad (\text{B10})$$

At separations $\ell \gg \ell_\eta^\perp$ where the direct effects of the resistive term $\sqrt{2\lambda} \boldsymbol{\eta}(t)$ can be neglected, the distance $\ell_\perp(t) = \sqrt{\langle y^2(t) \rangle}$ that field-lines advect apart in the field-perpendicular direction is obtained from

$$\frac{d}{dt} \ell_\perp^2 \sim D_\perp^e(\ell) \quad (\text{B11})$$

with

$$D_\perp^e(\ell) = \int_{-\infty}^0 dt \langle \delta \mathbf{u}_{e\perp}(\ell) \cdot \delta \mathbf{u}_{e\perp}(\ell, t) \rangle \quad (\text{B12})$$

The electron velocity increment in (B12) gets a contribution $\delta \mathbf{u}(\ell)$ from the bulk plasma velocity and another contribution $\delta_i \delta \mathbf{j}(\ell)$ from the Hall term. We would like to show that the Hall contribution is negligible whenever $\ell \gg \delta_i$. This can be argued as follows:

¹⁸ Note that energy flux to small scales is given in HMHD by

$$\Pi_\ell^E = -\boldsymbol{\tau}_\ell : \boldsymbol{\nabla} \bar{\mathbf{u}}_\ell - \boldsymbol{\varepsilon}_\ell \cdot \bar{\mathbf{j}}_\ell$$

and cross-helicity flux by

$$\Pi_\ell^C = -\boldsymbol{\tau}_\ell : [\boldsymbol{\nabla} \bar{\mathbf{b}}_\ell + \delta_i \boldsymbol{\nabla} \bar{\boldsymbol{\omega}}_\ell] - \bar{\boldsymbol{\omega}}_\ell \cdot \boldsymbol{\varepsilon}_\ell,$$

where $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{u}$ is the vorticity and $\boldsymbol{\tau}_\ell = \boldsymbol{\tau}_{R\ell} - \boldsymbol{\tau}_{M\ell}$ is the total stress, both Maxwell stress and Reynolds stress $\boldsymbol{\tau}_{R\ell} = \overline{(\mathbf{u}\mathbf{u})}_\ell -$

$\bar{\mathbf{u}}_\ell \bar{\mathbf{u}}_\ell$. At scales $\ell > \delta_i$, the terms without δ_i dominate and one gets constant energy and cross-helicity fluxes assuming GS95 scaling $\delta u(\ell) \sim \delta b(\ell) \sim (\varepsilon \ell_\perp^2)^{1/3}$.

¹⁹ For completeness we remark that the Ohmic field is also negligible with respect to $\boldsymbol{\varepsilon}_\ell^{Hall}$ whenever $\delta_i \delta b(\ell) \gtrsim \lambda$. This defines another resistive scale $\ell_\eta^{Hall\perp}$ which is greater or smaller than δ_i according as ℓ_η^\perp is greater or smaller than δ_i . In the latter case, $\ell_\eta^{Hall\perp}$ is the true, physical resistive scale, not ℓ_η^\perp .

First, we can estimate from energy balance alone without any other assumptions that $\varepsilon \sim \lambda j_{rms}^2$ and thus $j \sim j_{rms} = (\varepsilon/\lambda)^{1/2}$. This allows us compare the typical size of the Hall velocity with the flow velocity as

$$\frac{\delta_i j}{u} \sim \frac{\delta_i (\varepsilon/\lambda)^{1/2}}{u_L} = \frac{\delta_i}{\ell_\eta^\perp} \cdot S_L^{-1/4}, \quad (B13)$$

where we have used $\varepsilon = u_L^4/v_A L_i$ and we have defined $\ell_\eta^\perp \equiv (\lambda^3/\varepsilon)^{1/4}$, without the assumption that it plays any dynamical role as a “dissipation scale.” Now let the Lundquist number S_L increase by increasing L_i with ε fixed, and thus also λ, ℓ_η^\perp and the ratio

$$\alpha \equiv \delta_i/\ell_\eta^\perp \quad (B14)$$

all fixed as well. It follows from (B13) that $\delta_i j \ll u$ always when $\alpha < 1$ but even when $\alpha > 1$ if $S_L \gg \alpha^4$. We therefore expect that the Hall contribution to the 2-particle diffusivity is negligible compared with the advective MHD contribution at high Lundquist numbers.

Physically, the Hall term is very short-range correlated in both space and time, so that it acts on a pair of Lagrangian particles as two independent Brownian motions with an effective 1-particle diffusivity. This can be shown more formally, by observing that

$$\langle \delta \mathbf{j}_\perp(\ell) \cdot \delta \mathbf{j}_\perp(\ell, t) \rangle = 2[\langle \mathbf{j}_\perp(\mathbf{0}) \cdot \mathbf{j}_\perp(\mathbf{0}, t) \rangle - \langle \mathbf{j}_\perp(\mathbf{0}) \cdot \mathbf{j}_\perp(\ell, t) \rangle] \lesssim 2\langle \mathbf{j}_\perp(\mathbf{0}) \cdot \mathbf{j}_\perp(\mathbf{0}, t) \rangle, \quad (B15)$$

since the second term in the square brackets is a *decaying* term for increasing ℓ ²⁰. This upper bound is a considerable overestimate for smaller values of ℓ , but it suffices for our purpose. We can then estimate

$$\int_{-\infty}^0 dt \langle \mathbf{j}_\perp(\mathbf{0}) \cdot \mathbf{j}_\perp(\mathbf{0}, t) \rangle \sim j_{rms}^2 \tau_\lambda \quad (B16)$$

where $\tau_\lambda = (\lambda/\varepsilon)^{1/2} = 1/j_{rms}$ is the correlation time at the resistive dissipation scale. Multiplied by δ_i^2 , this is the Hall contribution to the 1-particle diffusivity. We therefore obtain a conservative upper bound on the Hall contribution to the 2-particle diffusivity, independent of ℓ , as

$$D_\perp^{Hall}(\ell) \lesssim \delta_i^2 j_{rms}^2 \tau_\lambda = \alpha^2 \lambda, \quad (B17)$$

where we have used (B13) and (B14).

Finally, we compare with the advective contribution for $\ell > \delta_i$. From the Eulerian argument presented earlier in the Appendix, we expect that GS95 scaling works on those length-scales. We can thus estimate for $\ell_\perp > \delta_i$ that

$$D_\perp^{MHD}(\ell) \sim \ell_\perp \delta u(\ell) \sim (\varepsilon \ell_\perp^4)^{1/3} \sim \alpha^{4/3} \lambda \left(\frac{\ell_\perp}{\delta_i} \right)^{4/3} \quad (B18)$$

where we have used $\lambda = (\varepsilon \ell_\eta^{\perp 4})^{1/3}$ and the definition of α . We conclude that

$$D_\perp^{Hall}(\ell)/D_\perp^{MHD}(\ell) \lesssim \alpha^{2/3} \left(\frac{\delta_i}{\ell_\perp} \right)^{4/3} \ll 1 \quad (B19)$$

at least when $\ell_\perp \gg \sqrt{\alpha} \delta_i$. Thus, Richardson diffusion of field-lines in HMHD turbulence is exactly the same as for MHD turbulence at perpendicular separations ℓ_\perp much greater than the ion skin depth δ_i .

Now let us reconsider the problem of *large-scale* reconnection treated in section 5, including the Hall effect within the HMHD model. Define $\Delta = L_i M_A^2$ taking $L_i \simeq L_x$. We shall make the essential assumption that $\delta_i \ll \Delta$. This assumption is almost always true in astrophysical reconnection problems. For example, using $\delta_i = \rho_i/\sqrt{\beta_i}$ and the data in Table 1, we can see that $\delta_i/\Delta \sim 10^{-13}$ for the warm ISM, $\sim 10^{-6}$ for post-CME current sheets and $\sim 10^{-1}$ for solar wind impinging on the magnetosphere. Only in the latter case is the condition not extremely well satisfied. Let us then repeat the arguments in section 5 taking into account the Hall term. Consider first the derivation of LV99 based on the stochastic wandering of magnetic field-lines as one traces points of increasing arc-length s along the lines. At small initial separations, the Hall term is important and the rms transverse distance $\ell_\perp(s) = \sqrt{\langle y^2(s) \rangle}$ grows as $\ell_\perp(s) \sim (s^3/\delta_i L_i) M_A^4$ ²¹. As soon as neighboring lines have separated to $\ell_\perp \gtrsim \delta_i$, then their further separation is the same as in standard MHD turbulence without the Hall term. However, to reach the separation $\sim \delta_i$ one must move along the field-lines a distance of only $s \sim L_i^* \equiv L_i (\frac{\delta_i}{\Delta})^{2/3} \ll L_i$ when $\delta_i \ll \Delta$. In that case, the formula (63) in section 5.1 for $\langle y^2(s) \rangle$ still holds, if $s \gg L_i^*$. It therefore follows that the LV99 estimate for the width of the reconnection layer and the formula (64) for the reconnection velocity will hold, as long as $\delta_i \ll \Delta$, since most of the separation of lines will be achieved for $s \geq L_i^*$. The same conclusion can be derived from the Lagrangian perspective.

²⁰ Note that $\langle j_i(\mathbf{r}) j_k(\mathbf{0}) \rangle \sim \Delta \langle \delta b_i(\mathbf{r}) \delta b_k(\mathbf{r}) \rangle \sim r^{2(h_B-1)}$ decays for large r whenever the magnetic-field increment $\delta \mathbf{b}(\mathbf{r})$ has scaling exponent $h_B < 1$.

²¹ Obtained by solving $d\ell_\perp/ds \sim db(\ell)/v_A = (1/v_A)(\varepsilon \ell_\perp^2/\delta_i)^{1/3}$.

For small initial separations $\ell_\perp < \delta_i$ the lines will separate laterally as $\ell_\perp(t) \sim (\delta_i^2 \varepsilon t^3)^{1/4}$ ²². However, after a time $t \sim t_c^* = (\delta_i^2/\varepsilon)^{1/3} = (\frac{\delta_i}{\Delta})^{2/3} (L_i/v_A)$ much shorter than $t_c = L_i/v_A$ the pairs of lines will reach the separation $\ell_\perp(t) \sim \delta_i$ and GS95 scaling will take over. The formula (60) for $\langle y^2(t) \rangle$ that follows from GS95 will hold as soon as $t > t_c^*$ and the bulk of the separation of lines will occur after this time if $\delta_i \ll \Delta$. Thus, the estimate (62) for the width of the reconnection layer holds just as before, with only a negligible correction due to finite δ_i .

We thus reach the important conclusion that *the Hall term is irrelevant to determining the rate of global reconnection in the presence of high-Reynolds-number plasma turbulence*. At most a modest enhancement of the reconnection rate is obtained, but vanishingly small when $\delta_i \ll \Delta$. We should mention that a similar conclusion was reached by Smith et al. (2004) on the basis of numerical simulations. However, that study can be fairly criticized because the simulations were performed in 2D and at such low Reynolds numbers that most of the measured reconnection rates (in a transient regime) were smaller than the steady-state Sweet-Parker rates. Our theoretical arguments, on the other hand, are valid in 3D and at arbitrarily high Reynolds numbers. This is not to say that the Hall term will have no effect whatsoever, but these will be confined to very small scales, e.g. modifying the structure of local reconnection events. However, these local effects are irrelevant to determining the global rate of large-scale reconnection.

Similar conclusions can be reached not only for the 2-fluid Hall term, but also for other microscopic plasma mechanisms proposed to enhance reconnection speeds. For example, the MHD turbulent cascade may be terminated not by Spitzer resistivity but instead by some form of anomalous resistivity, e.g. due to ion acoustic instability. The latter produces a large effective magnetic diffusivity of order $(m_i/m_e)^{1/2} \delta_e^2 \omega_{c,e}$ at scales smaller than $\delta_i/\sqrt{\beta}$ (e.g. see Galeev & Sagdeev (1984) and Kulsrud (2005), section 14.7.) But larger scales will behave as standard MHD turbulence and the LV99 predictions will again hold as long as $\delta_i \ll \Delta$. This was verified in the simulations of Kowal et al. (2009), who found no effect of anomalous resistivity on reconnection speed. Turbulence is such a powerful accelerator of magnetic line-separation that it completely dominates microscopic plasma processes of line diffusion, as soon as the magnetic and kinetic Reynolds numbers are sufficiently high.

²² Obtained by solving $d\ell_\perp/dt = \delta_i \delta b(\ell)/\ell_\perp = (\varepsilon \delta_i^2/\ell_\perp)^{1/3}$

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